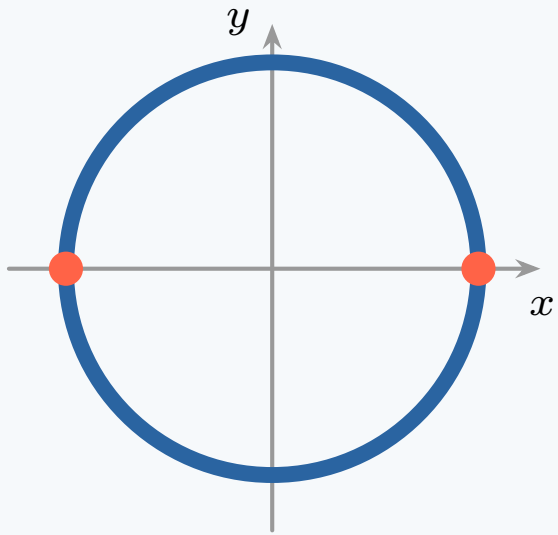
What is...perfectoid spaces?

Or: How can two kinds of arithmetic share one geometry?

One equation can live in many worlds

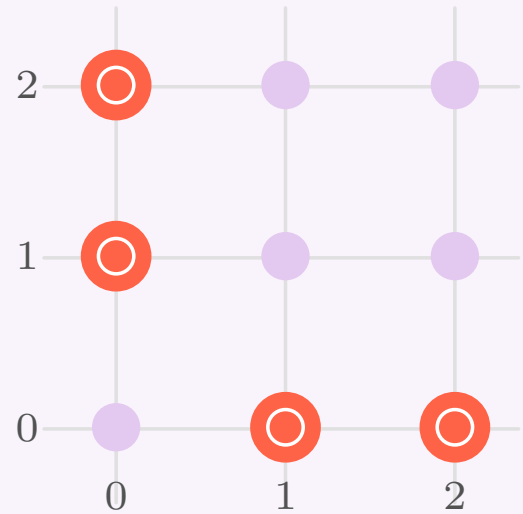
$$x^2 + y^2 = 1$$

Over the real numbers



a continuous curve

Modulo 3

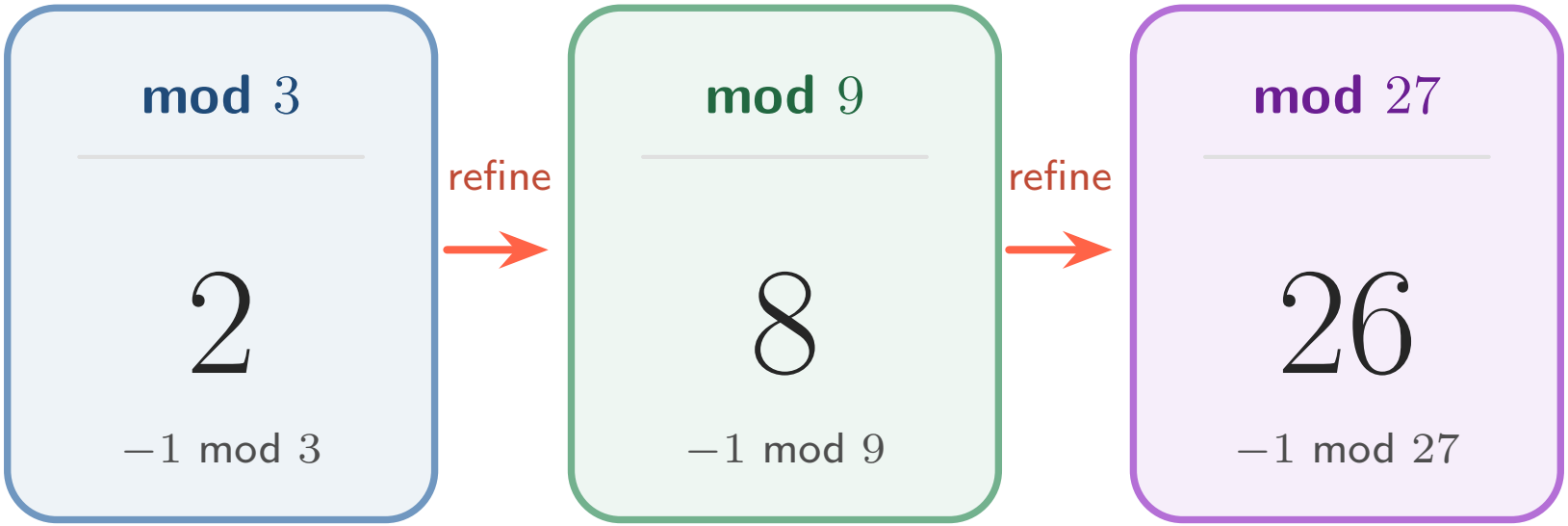


exactly four solutions

- Number theory Find rational solutions of polynomial equations
- Example A circle over $\mathbb{R}$, but only four points modulo 3
- Big picture Relate these two worlds in some way or form
- Here and below, 3 can be replaced by any prime

1897: keep all the reductions at once

One integer, seen with increasing 3-adic precision

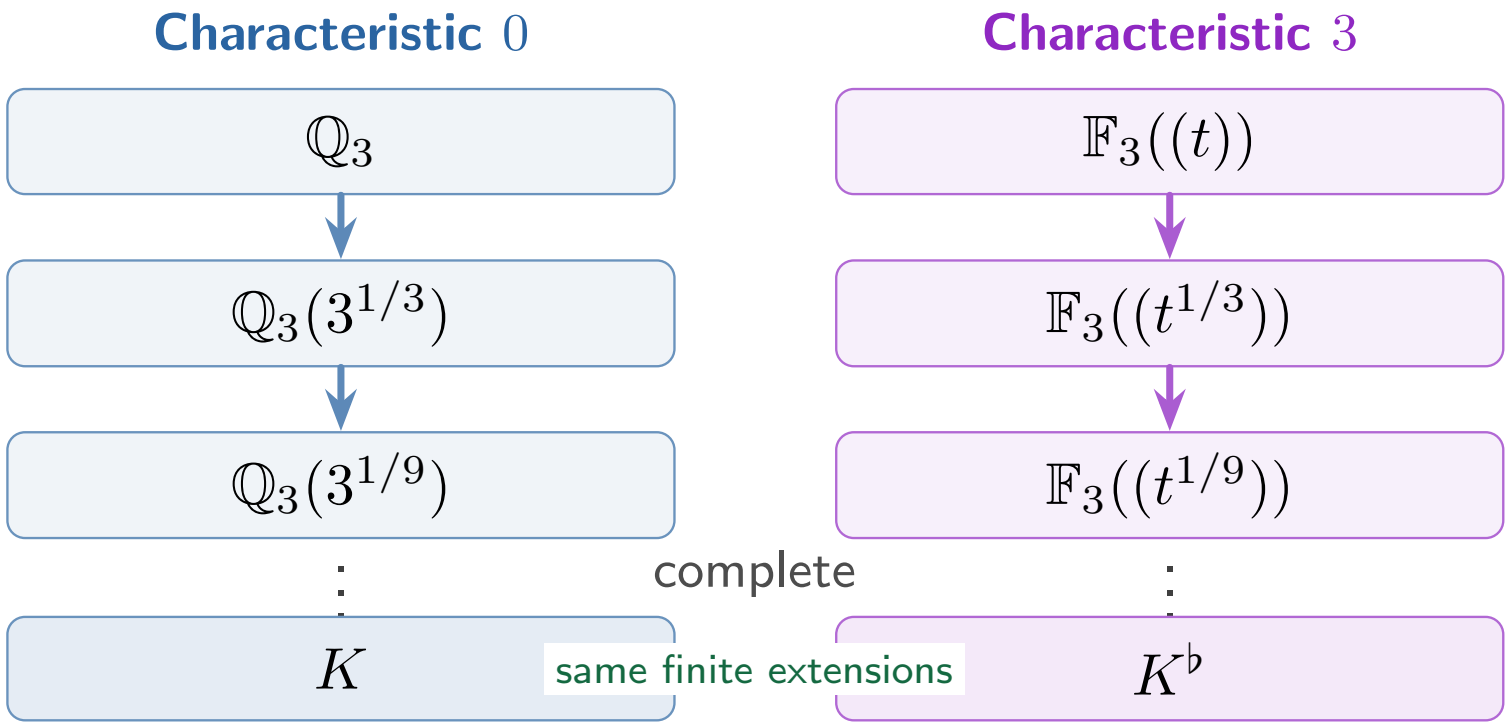


$$\cdots 222_3 = -1$$

An infinite, compatible list of remainders is one 3-adic integer

- Modulo 3 Easy to compute, but most information is lost
- Hensel's idea Keep the data modulo $3, 9, 27, \ldots$ simultaneously
- New geometry Study equations over the 3-adic numbers

1979: Galois theory crosses characteristics

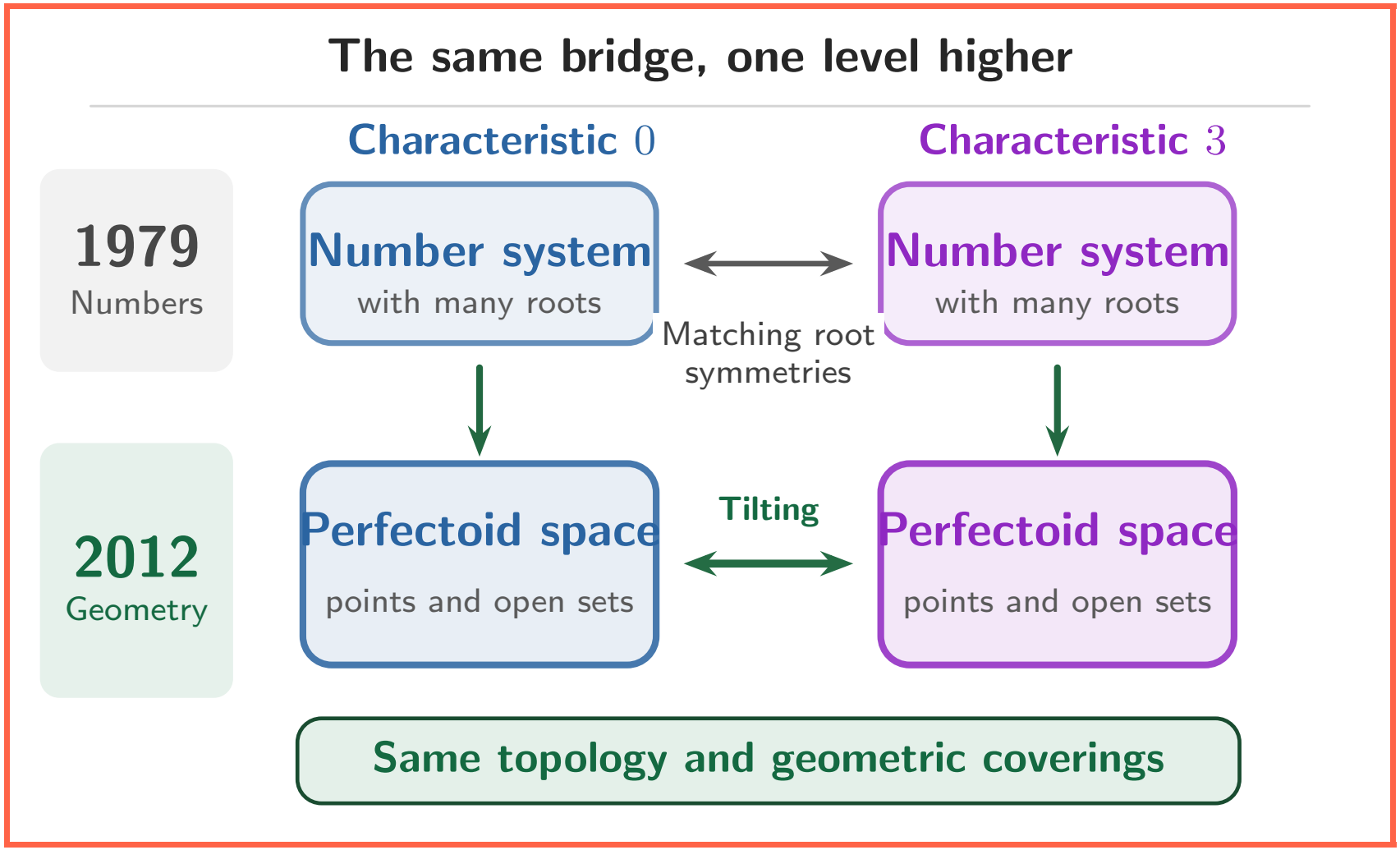


Fontaine–Wintenberger: an isomorphism of absolute Galois groups

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \;\cong\; \mathrm{Gal}\big((K^\flat)^{\mathrm{sep}}/K^\flat\big)$$

- Two towers Adjoin roots of 3 and t, and then complete
- Different fields The limits $K, K^\flat$ have characteristics $0, 3$
- The surprise Their finite field extensions correspond

2012: from numbers to geometry



- The idea Instead of comparing numbers, compare entire geometric spaces
- Tilting Certain spaces in characteristic 0 have partners in characteristic 3
- The miracle Their points, topology and geometric coverings correspond

Why cross the bridge at all?

A tiny example of why characteristic 3 can be easier

Characteristic 0

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

extra terms in orange

$3 = 0$ →

Characteristic 3

$$(a+b)^3 = a^3 + b^3$$

the extra terms vanish

Tilting lets some geometry use this simpler arithmetic.

- Simpler arithmetic In characteristic 3, cubing a sum is much easier
- The strategy Solve a suitable problem on the easier side, then transfer back
- The payoff Scholze proved new cases of a 40-year-old conjecture for shapes cut out by one equation

Thank you for your attention!

Next time: Prismatic cohomology