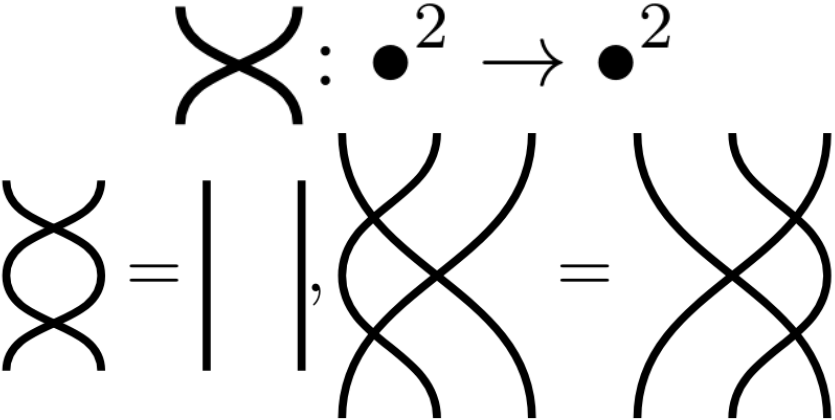
What is...diagrammatic algebra?

Or: Subfields of mathematics 17

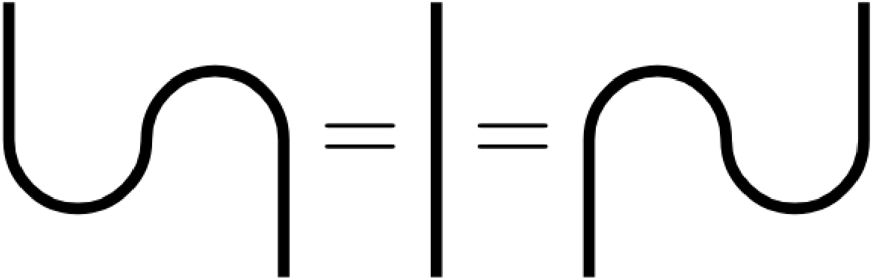
Swapping factors



- Say we have some set $\bullet$; consider $\bullet^d = \bullet \times \ldots \times \bullet$
- Take the swap map: $\bullet^2 \to \bullet^2, (x, y) \mapsto (y, x)$
- Above The diagrammatic algebra of the swap map

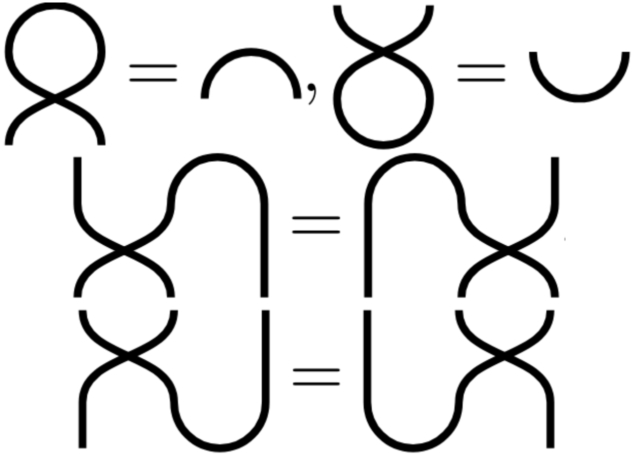
Pairing factors

$$\cap : \bullet^2 \to \mathbb{1}, \cup : \mathbb{1} \to \bullet^2$$



- Say we have some $\mathbb{C} = \mathbb{1}$ vector space $\bullet$; consider $\bullet^d = \bullet \otimes \ldots \otimes \bullet$
- Take a (nondegenerate symmetric) pairing map: $\bullet^2 \to \mathbb{1}, x \otimes y \mapsto \langle x, y \rangle$ (identify $\bullet$ with its dual); take the copairing $\mathbb{1} \to \bullet^2, 1 \mapsto \sum_{basis} x \otimes x^*$
- Above The diagrammatic algebra of the (co)pairing maps

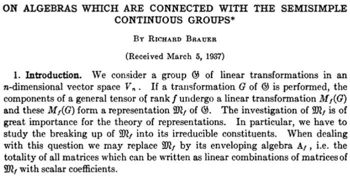
Swapping and pairing



- Say we have some $\mathbb{C} = \mathbb{1}$ vector space $\bullet$; consider $\bullet^{d} = \bullet \otimes \ldots \otimes \bullet$
- Take swapping and pairing together
- Above The diagrammatic algebra of the swapping and (co)pairing maps *Br*

Enter, the theorem

Br gives diagrammatics of orthogonal $O_n(\mathbb{C})$ or symplectic $SP_n(\mathbb{C})$ invariants

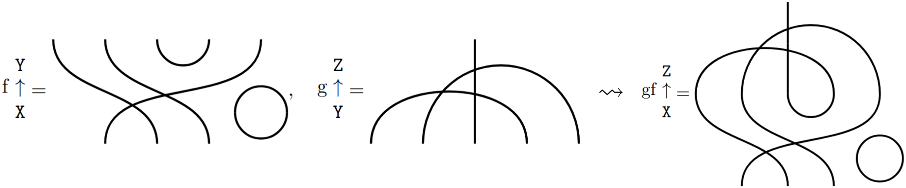

ON ALGEBRAS WHICH ARE CONNECTED WITH THE SEMISIMPLE CONTINUOUS GROUPS*

BY RICHARD BRAUER

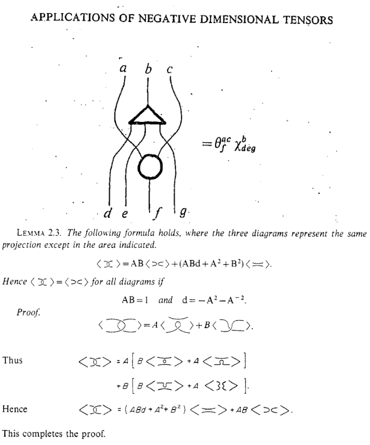
(Received March 5, 1937)

1. **Introduction.** We consider a group $\mathfrak{G}$ of linear transformations in an n-dimensional vector space V_n. If a transformation G of $\mathfrak{G}$ is performed, the components of a general tensor of rank f undergo a linear transformation $M_f(G)$ and these $M_f(G)$ form a representation $\mathfrak{M}_f$ of $\mathfrak{G}$. The investigation of $\mathfrak{M}_f$ is of great importance for the theory of representations. In particular, we have to study the breaking up of $\mathfrak{M}_f$ into its irreducible constituents. When dealing with this question we may replace $\mathfrak{M}_f$ by its enveloping algebra A_f, i.e. the totality of all matrices which can be written as linear combinations of matrices of $\mathfrak{M}_f$ with scalar coefficients.

- **Formally** There is a full and essentially surjective functor $Br \to Rep(O)$ or $Rep(SP)$, and the 'kernel' can also be described
- **Comment** $GL_n(\mathbb{C})$ does not have a natural pairing, hence $O_n(\mathbb{C})$, $SP_n(\mathbb{C})$
- Diagrammatic algebra answers similar questions!



Diagrams are everywhere


APPLICATIONS OF NEGATIVE DIMENSIONAL TENSORS

LEMMA 2.3. *The following formula holds, where the three diagrams represent the same projection except in the area indicated.*

$$\langle \; \rangle = AB\langle \; \rangle + (ABd + A^2 + B^2)\langle \; \rangle.$$

Hence $\langle \; \rangle = \langle \; \rangle$ *for all diagrams if*

$$AB = 1 \quad \textit{and} \quad d = -A^2 - A^{-2}.$$

Proof.

$$\langle \; \rangle = A\langle \; \rangle + B\langle \; \rangle.$$

Thus $\langle \; \rangle = A\left[B\langle \; \rangle + A\langle \; \rangle\right] + B\left[B\langle \; \rangle + A\langle \; \rangle\right].$

Hence $\langle \; \rangle = (ABd + A^2 + B^2)\langle \; \rangle + AB\langle \; \rangle.$

This completes the proof.

- Diagrammatic algebra seems to originate in 19th century invariant theory
- However, it took much longer to fly
- 'Modern' examples Penrose/Feynman diagrams, quantum topology ...

Thank you for your attention!

I hope that was of some help.