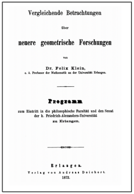
What is...invariant theory?

Or: Subfields of mathematics 18

Group actions rule!

The Erlangen Programme

"Given a [homogeneous] manifold and a transformation group acting [transitively] on it, to investigate those properties of figures on that manifold which are invariant under transformations of that group"



Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

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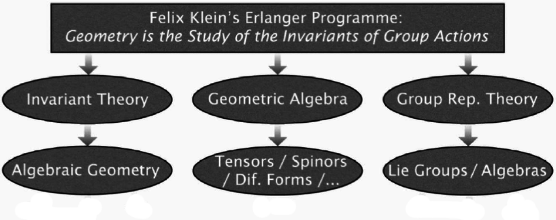
Programm

zum Eintritt in die philosophische Facultät und den Senat der k. Friedrich-Alexanders-Universität zu Erlangen.

Erlangen.

Verlag von Andreas Deichert.

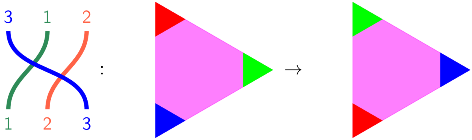
1872.

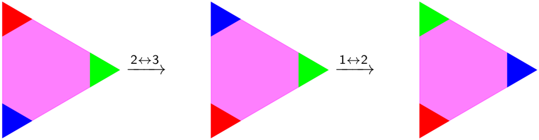


- **Above** The Erlangen program
- **Slogan** Group actions describe many properties
- Several fields originate or habe **connections to** the Erlangen program

Symmetries of simplices

The symmetric group in three letters acts on a triangle via the rule "green=1, red=2, blue=3, and then permute":





- $S_n = \mathrm{Aut}(\{1, ..., n\})$, the symmetric group, acts on a simplex
- S_n determines the symmetries of a simplex
- Invariant theory question What are the fixed points? "Invariant ⇝ fixed"

Invariant polynomials

For $n = 1$:

$$e_1(X_1) = X_1.$$

For $n = 2$:

$$e_1(X_1, X_2) = X_1 + X_2,$$
$$e_2(X_1, X_2) = X_1 X_2.$$

For $n = 3$:

$$e_1(X_1, X_2, X_3) = X_1 + X_2 + X_3,$$
$$e_2(X_1, X_2, X_3) = X_1 X_2 + X_1 X_3 + X_2 X_3,$$
$$e_3(X_1, X_2, X_3) = X_1 X_2 X_3.$$

For $n = 4$:

$$e_1(X_1, X_2, X_3, X_4) = X_1 + X_2 + X_3 + X_4,$$
$$e_2(X_1, X_2, X_3, X_4) = X_1 X_2 + X_1 X_3 + X_1 X_4 + X_2 X_3 + X_2 X_4 + X_3 X_4,$$
$$e_3(X_1, X_2, X_3, X_4) = X_1 X_2 X_3 + X_1 X_2 X_4 + X_1 X_3 X_4 + X_2 X_3 X_4,$$
$$e_4(X_1, X_2, X_3, X_4) = X_1 X_2 X_3 X_4.$$

- Extend the previous action linearly $\Rightarrow$ fixed point $x_1 + \ldots + x_n$
- $x_1 + \ldots + x_n$ spans the degree one invariants
- Idea Why not extend the action to the whole polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$

Enter, the theorem

Let $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ denote the invariants under the previous action

(i) $\mathbb{C}[x_1, \dots, x_n]^{S_n} =$ symmetric polynomials

(ii) Symmetric polynomials are spanned by elementary symmetric polynomials e_k

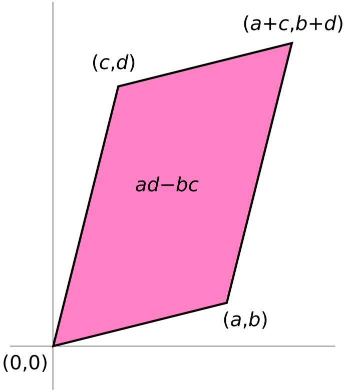
(iii) The e_k are algebraically independent

(iv) $\mathbb{C}[x_1, \dots, x_n]^{S_n} \cong \mathbb{C}[e_1, \dots, e_n]$ Another polynomial ring

- This is a bit like:

- Invariant theory answers similar questions!

A more standard example



- Above The (abs of the) determinant is the area of the parallelogram
- Classical question What are the invariants of $SL_n(\mathbb{C})$ acting on n-by-n matrices by left multiplication?
- Answer $\mathbb{C}[Mat_n]^{SL_n(\mathbb{C})} \cong \mathbb{C}[det]$

Thank you for your attention!

I hope that was of some help.