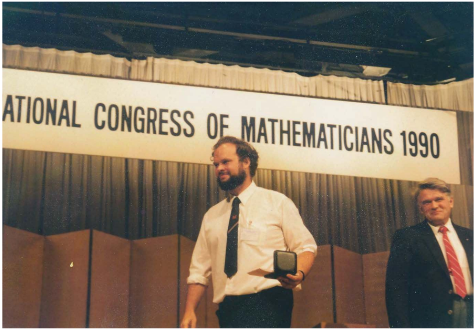
What is...quantum algebra?

Or: Subfields of mathematics 21

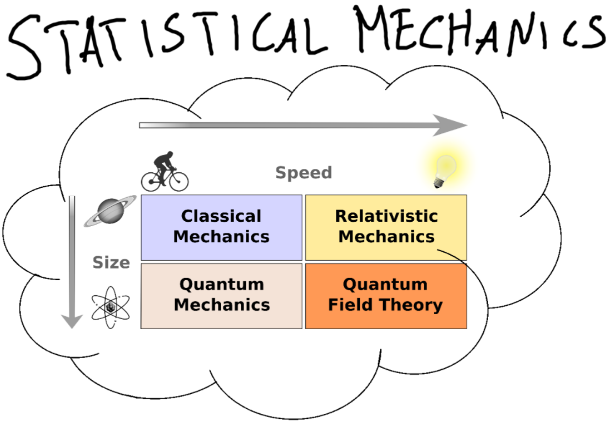
The birth of quantum ~~topology~~ algebra

Jones was awarded the Fields Medal at Kyoto in 1990 for these breakthroughs.



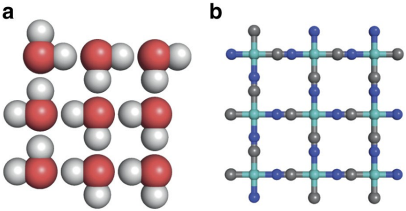
- Kyoto 1990 Jones receives the fields medal ⇒ quantum topology
- To the right Faddeev, a key figure of quantum algebra which was discovered, essentially independently, around the same time
- Today A brief, incomplete, and mostly wrong history of quantum groups

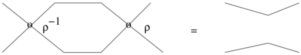
Statistical mechanics



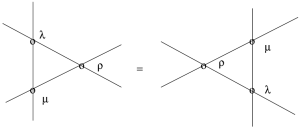
- Statistical mechanics = study large assemblies e.g. probabilistically
- Key Its methods are applied to many fields of physics and beyond
- Example Box of gas: precise position of a particle is impossible to get, but global behavior is easy to predict

Ice models and the R matrix





Yang–Baxter equations (YBE):



- While studying ice models, there was a need for an operator (=matrix), called crossing or R matrix, satisfying the above YBE
- First nontrivial case Each string is a 2d space, so the matrix should be 4-by-4

Enter, the theorem

A solution to the YBE is:

$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \begin{pmatrix} zq^{-1} - z^{-1}q & 0 & 0 & 0 \\ 0 & z^{-1}(q^{-1} - q) & z - z^{-1} & 0 \\ 0 & z - z^{-1} & z(q^{-1} - q) & 0 \\ 0 & 0 & 0 & zq^{-1} - z^{-1}q \end{pmatrix}$$

$$z = \exp(-\lambda), q = \exp(\theta)$$

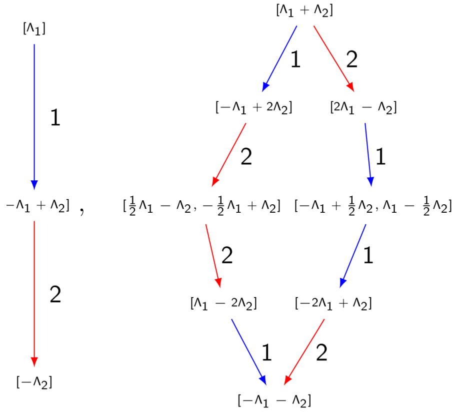
$$R = \lim_{z \to 0} R(z)$$

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2 - 1 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

q is the quantum parameter ($q = 1$ is nonquantum), and z the spectral parameter

- After the discovery of this solution, people ask: Where does this come from?
- In the process of answering this, quantum groups were discovered
- Example The above solution comes from $U_q(\mathfrak{sl}_2)$ or $U_q(\hat{\mathfrak{sl}}_2)$ (with z)
- Quantum algebra studies similar "quantum stuff"!

Two key achievements of quantum algebra



- **Key achievement 1** The Jones-type-invariants are all quantum
- **Key achievement 2** The substitution $q=0$ or $q=\infty$ makes sense, and gives combinatorial models of Lie algebras and their representations
- **$q=0$ or $q=\infty$** This is called the crystal limit

Thank you for your attention!

I hope that was of some help.