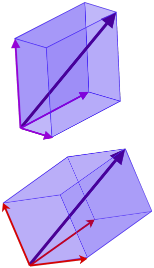
What is...matroid theory?

Or: Subfields of mathematics 23

Fundamental properties of bases

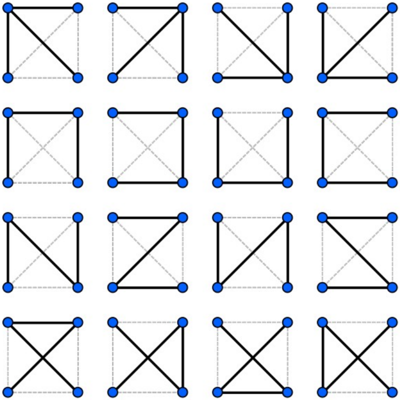


$$, \quad \mathbf{A} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$B_1 = \left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix} \right\}, \quad B_3 = \left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix} \right\}$$

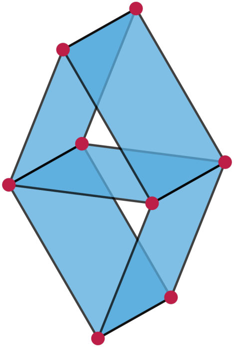
- **Question** What makes bases of vector spaces special?
- **Answer attempt 1** They exist!
- **Answer attempt 2** We can always exchange vectors between them without loosing the property of being a basis

Fundamental properties of forests



- Question What makes spanning forests of graphs special?
- Answer attempt 1 They exist!
- Answer attempt 2 We can always exchange edges between them without loosing the property of being a spanning forest

Matroids



- A matroid is a pair $(E, \mathfrak{B})$ of a finite set E and bases $\mathfrak{B} \subset \mathfrak{P}(E)$ such that:
  - (i) $\mathfrak{B}$ is not empty Existence of bases
  - (ii) For $A \neq B$ in $\mathfrak{B}$ and $a \in A \setminus B$ there exists $b \in B$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathfrak{B}$ Basis exchange property
- Above Take eight points and bases = collection of four points which are not the ones in the picture

Enter, the theorem

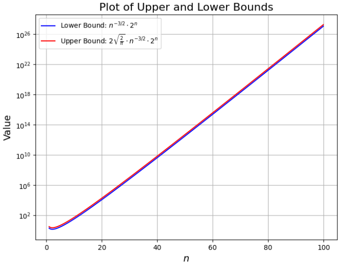
There are many matroids; m_n = number of simple matroids of rank n, then:

(i) First few values are 1, 1, 1, 2, 4, 9, 26, 101, 950, 376467

A002773 Number of nonisomorphic simple matroids (or geometries) with n points.
(Formerly M1197 N0462)

1, 1, 1, 2, 4, 9, 26, 101, 950, 376467

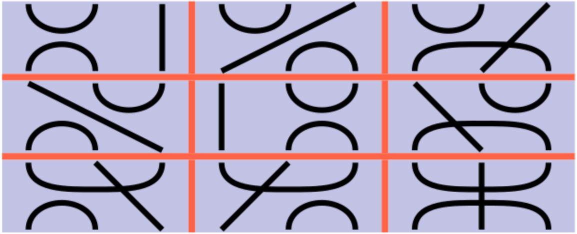
(ii) $n^{-3/2} \cdot 2^n \leq \log m_n \leq 2\sqrt{2/\pi} \cdot n^{-3/2} \cdot 2^n \cdot (1 + o(1))$

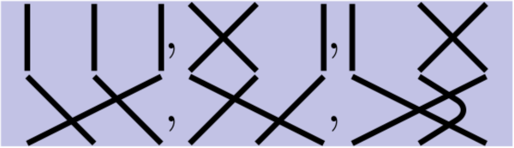


- Rank = number of basis elements
- Matroid theory answers similar questions!

Not linear algebra, not graph theory, but more

2-partitions of $\{1, 2, 3, 1', 2', 3'\}$:





- **Question** What makes 2-partitions of a set special?
- **Answer attempt 1** They exist!
- **Answer attempt 2** We can always exchange elements between them without loosing the property of being a 2-partition

Thank you for your attention!

I hope that was of some help.