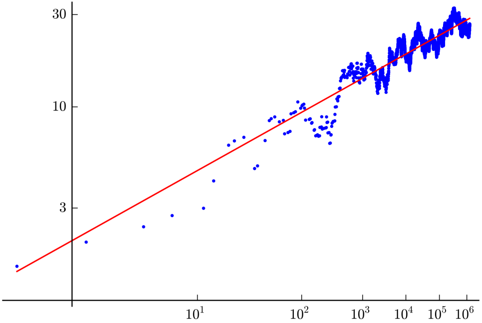
What is...automated mathematics?

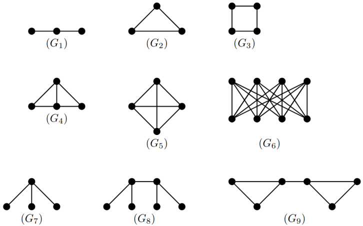
Or: Subfields of mathematics 24

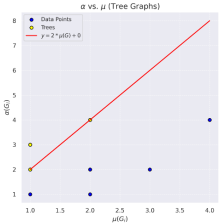
No humans, please!



- Mathematics is, at least partially, about good conjectures
- Computers are nowadays key for the art of conjecturing
- Early example The Birch–Swinnerton-Dyer conjecture was discovered by computer
- There are 3 stages of conjecturing: computer assisted, AI assisted and, as in this video, without humans

Stage 3: Example



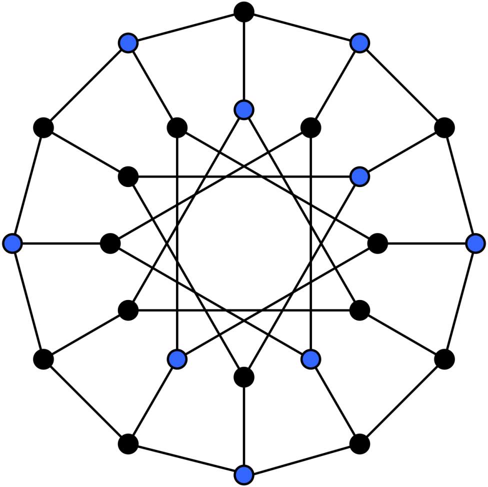


name	n	μ	α	$n-\mu$	$n-\delta$	Δ^2	connected	tree	regular	bipartite
G_1	3	1	2	2	2	4	True	True	False	True
G_2	3	1	1	2	1	4	True	False	True	False
G_3	4	2	2	2	2	4	True	False	True	True
G_4	4	2	2	2	2	9	True	False	False	False
G_5	4	2	1	2	1	9	True	False	True	False
G_6	8	4	4	4	4	16	True	False	True	True
G_7	4	1	3	3	3	9	True	True	False	True
G_8	6	2	4	4	5	9	True	True	False	True
G_9	6	3	2	3	4	9	True	False	False	False

- Stage 3 Automated conjecturing
- Graffiti (a program that knows certain graphs and graph properties, ∼1985) creates conjectures by data search, trying to match graph+property
- Bait-and-catch No human input at all, but the setting is very restricted and almost all conjectures are rather boring

A reminder from graph theory

$\alpha(G) = 9$:



- $\alpha(G)$ = size of a maximum independent vertex set
- $\mu(G)$ = size of a maximum independent edge set

Enter, the theorem

It is impressive what Graffiti and follow-ups conjectured, and a lot of it was proven, e.g.:

(i) Example conjecture and proof

Listing 7 Example Conjecture

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Conjecture 9. If G is connected and regular, then matching_number(G) >=
    independence_number(G). This bound is sharp on 3 graphs.
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Theorem 1 (Caro et al. [64]). *If G is an r-regular graph with $r > 0$, then*

$$\alpha(G) \leq \mu(G),$$

and this bound is sharp.

(ii) More conjectures and proofs

Conjecture	Graph Family	Authors and Publication
$\alpha(G) \leq \mu(G)$	regular graphs	Caro et al. [64]
$Z(G) \leq \beta(G)$	claw-free graphs	Brimkov et al. [65]
$\alpha(G) \leq \frac{3}{2}\gamma_t(G)$	cubic graphs	Caro et al. [66]
$\alpha(G) \leq \gamma_2(G)$	claw-free graphs	Caro et al. [66]
$\gamma_e(G) \geq \frac{3}{5}\mu(G)$	cubic graphs	Caro et al. [66]
$Z(G) \leq 2\gamma(G)$	cubic graphs	Davila and Henning [67]
$Z_t(G) \leq \frac{3}{2}\gamma_t(G)$	cubic graphs	Davila and Henning [68]
$Z(G) \leq \gamma(G) + 2$	cubic claw-free graphs	Davila [69]

Table 2 Notable conjectures in graph theory generated by *TxGraffiti* and their corresponding publications.

- Automated mathematics = no humans ☺(for theorems, conjectures, ...)
- Automated mathematics answers similar questions!

Not just graph theory

C.E. Larson, N. Van Cleemput / Artificial Intelligence 231 (2016) 17–38

Table 2
Upper bound conjectures for the determinant of a symmetric matrix.

1.	det(x)	≤	permanent(x)
2.	det(x)	≤	minimum_eigenvalue(x)*trace(x)
3.	det(x)	≤	maximum_eigenvalue(x)*trace(x)
4.	det(x)	≤	(rank(x) + 1)*spectral_radius(x)
5.	det(x)	≤	permanent(x)+max_column_sum(x)+1
6.	det(x)	≤	maximum(rank(x), minimum_eigenvalue(x)^2)
7.	det(x)	≤	maximum_eigenvalue(x)*minimum(minimum_eigenvalue(x), trace(x) + 1)
8.	det(x)	≤	minimum_eigenvalue(x)*minimum(trace(x), maximum_eigenvalue(x))
9.	det(x)	≤	maximum_eigenvalue(x)^1_inf_norm(x) + separator(x)
10.	det(x)	≤	trace(x)*average_eigenvalue(x) - permanent(x)
11.	det(x)	≤	(maximum_eigenvalue(x)+1)*minimum_eigenvalue(x)+frobenius_norm(x)

Table 3
Lower bound conjectures for the determinant of a symmetric matrix.

1.	det(x)	≥	minimum_eigenvalue(x)*separator(x)
2.	det(x)	≥	minimum(permanent(x), log(nullity(x)))
3.	det(x)	≥	-2*1_inf_norm(x)^nrows(x) + permanent(x)
4.	det(x)	≥	-(separator(x) - 1)*frobenius_norm(x) + permanent(x)
5.	det(x)	≥	-1_inf_norm(x)*frobenius_norm(x)
6.	det(x)	≥	minimum(rank(x)-1, minimum_eigenvalue(x)/nullity(x))
7.	det(x)	≥	-4*1_inf_norm(x)^2 + permanent(x)

- The same strategy has been applied in many fields
- Example above Conjectures about matrices
- Missing This method gives also many ‘boring’ conjectures – its a bit ‘test all’ instead fo something smarter – unclear how to fix this in 2024

Thank you for your attention!

I hope that was of some help.