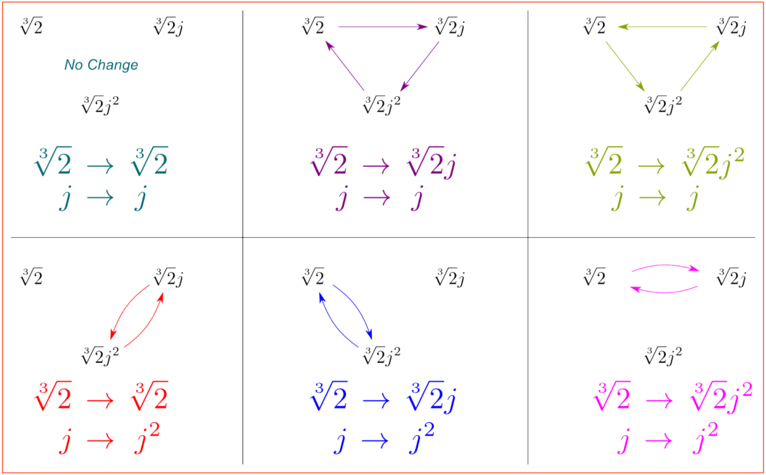
What is...Lie theory?

Or: Subfields of mathematics 26

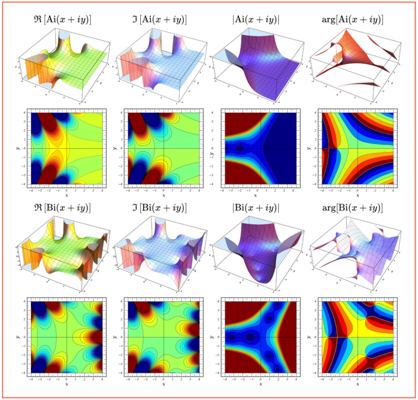
The discrete groups



- Galois theory = study of symmetries of polynomial equations $p(x) = 0$
- The appearing groups are discrete/finite groups like symmetric groups
- Idea The difficulty of the group $\leftrightsquigarrow$ difficulty of finding solutions to $p(x) = 0$

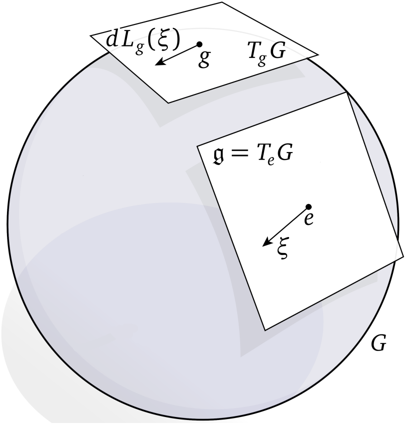
The continuous groups

Airy function:



- Lie theory = study of symmetries of differential equations $p(x) = 0$
- The appearing groups are continuous groups like $SL_2(\mathbb{C})$
- Idea The difficulty of the group $\rightsquigarrow$ difficulty of finding solutions to $p(x) = 0$
- Example The Galois-type group of the Airy equation $\frac{d^2y}{dx^2} - xy = 0$ is $SL_2(\mathbb{C})$

From continuous to linear



- **Observation** The continuous groups (call them Lie groups G) are actually easier than the discrete groups
- **Observation** The Lie groups are smooth manifolds
- **Idea** Study their tangent space (call them Lie algebras $\mathfrak{g}$) - these are now linear objects and part of linear algebra

Enter, the theorem

Lie group-Lie algebra correspondence Say our ground field is $\mathbb{R}$ (char. 0)

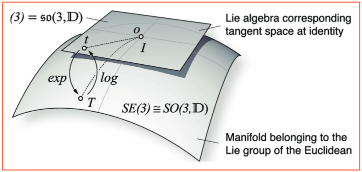
(i) Every Lie group G has an associated Lie algebra $\mathfrak{g}$

(ii) $G \cong H$ as Lie groups $\Rightarrow$ $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras

(iii) G, H are simply connected: $G \cong H$ as Lie groups $\Leftrightarrow$ $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras

Thus, Lie groups and Lie algebras are essentially the same

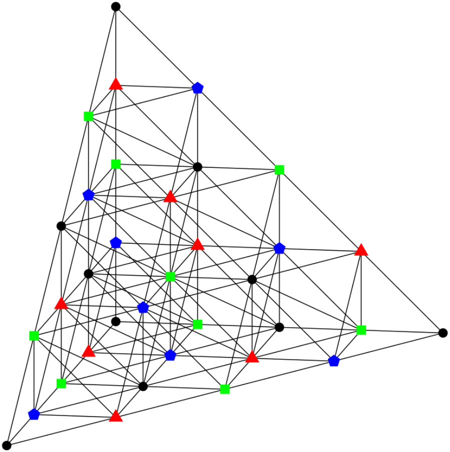
- Even better The representation theory of G and $\mathfrak{g}$ are also essentially the same
- The maps between them are exp and log



- Lie theory answers similar questions!

Its even 'just' combinatorics

$SL_4(\mathbb{C})$ weights:



- Moral Symmetries differential equations are determined by linear objects
- Even better Lie algebras are essentially determined by root/weight systems
- These are very rigid and almost purely combinatorial in nature

Thank you for your attention!

I hope that was of some help.