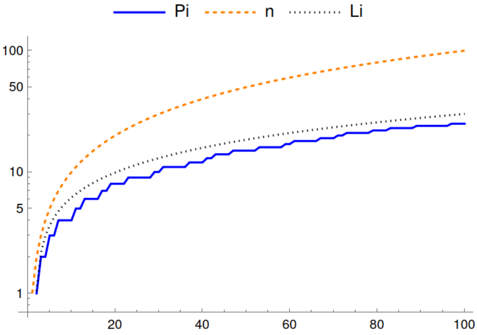
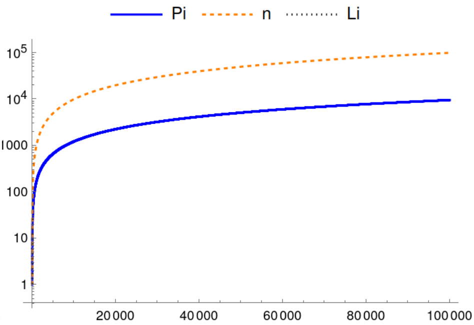
What is...analytic number theory?

Or: Subfields of mathematics 4

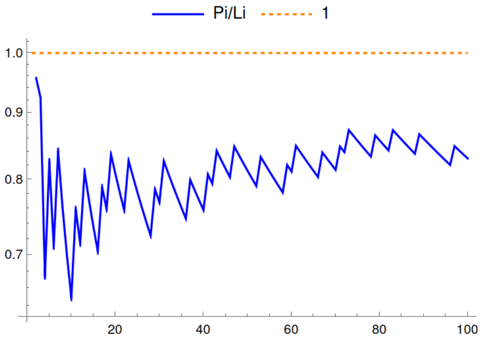
The art of not counting primes I

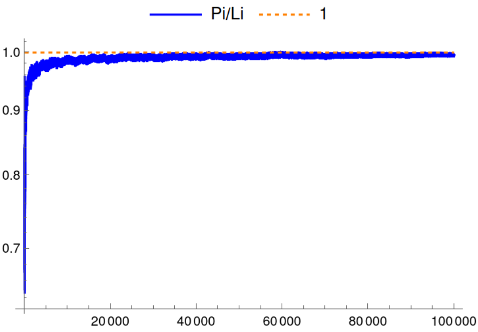




- Prime number function $\pi(n)$ = # primes $\leq$ n
- Counting primes is very tricky as primes “pop up randomly”
- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

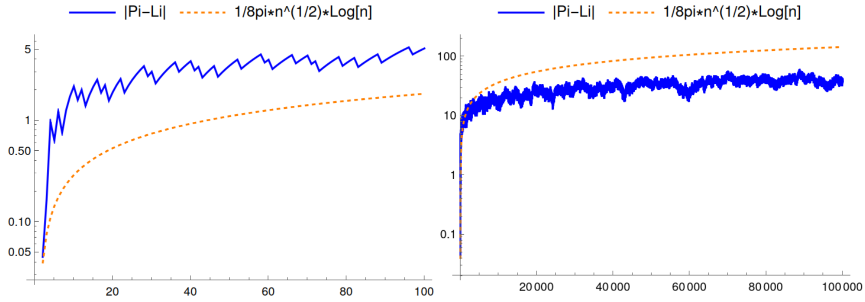
The art of not counting primes II





- Asymptotically equal $f \sim g$ if $\lim_{n \to \infty} f(n)/g(n) \to 1$
- Logarithmic integral $Li(x) = \int_2^x 1/\ln(t)dt$
- Question 2 What is the growth (of the number of primes) asymptotically?
- Answer 2 We have $\pi(n) \sim n/\log(n) \sim Li(n)$

The art of not counting primes III



- Asymptotically equal does not imply that the difference is good
- $|f(n) - g(n)|$ is a measurement of how good the approximation is
- Question 3 What is variance from the expected value ($Li(n)$)?
- Conjectural answer 3 We have $|\pi(n) - Li(n)| \in O(n^{1/2} \log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)

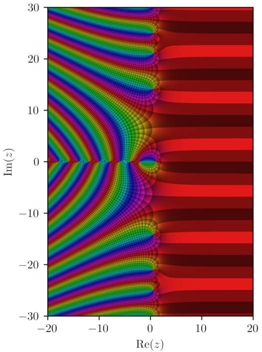
Enter, the theorem

For a and d coprime there are ∞ many primes of the form $a + nd$

Arithmetic progression	First 10 of infinitely many primes	OEIS sequence
$2n + 1$	3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...	A065091
$4n + 1$	5, 13, 17, 29, 37, 41, 53, 61, 73, 89, ...	A002144
$4n + 3$	3, 7, 11, 19, 23, 31, 43, 47, 59, 67, ...	A002145
$6n + 1$	7, 13, 19, 31, 37, 43, 61, 67, 73, 79, ...	A002476
$6n + 5$	5, 11, 17, 23, 29, 41, 47, 53, 59, 71, ...	A007528
$8n + 1$	17, 41, 73, 89, 97, 113, 137, 193, 233, 241, ...	A007519
$8n + 3$	3, 11, 19, 43, 59, 67, 83, 107, 131, 139, ...	A007520
$8n + 5$	5, 13, 29, 37, 53, 61, 101, 109, 149, 157, ...	A007521
$8n + 7$	7, 23, 31, 47, 71, 79, 103, 127, 151, 167, ...	A007522
$10n + 1$	11, 31, 41, 61, 71, 101, 131, 151, 181, 191, ...	A030430
$10n + 3$	3, 13, 23, 43, 53, 73, 83, 103, 113, 163, ...	A030431
$10n + 7$	7, 17, 37, 47, 67, 97, 107, 127, 137, 157, ...	A030432
$10n + 9$	19, 29, 59, 79, 89, 109, 139, 149, 179, 199, ...	A030433
$12n + 1$	13, 37, 61, 73, 97, 109, 157, 181, 193, 229, ...	A068228
$12n + 5$	5, 17, 29, 41, 53, 89, 101, 113, 137, 149, ...	A040117
$12n + 7$	7, 19, 31, 43, 67, 79, 103, 127, 139, 151, ...	A068229
$12n + 11$	11, 23, 47, 59, 71, 83, 107, 131, 167, 179, ...	A068231

- Dirichlet's theorem on arithmetic progressions was (one of the first) discrete problems solved using analytic methods
- Analytic number theory helps to answer and answers similar questions!

Enter, analysis



- Dirichlet's proof uses so-called L-functions
- Example (above) The Riemann zeta function is an L-function
- Key in the proof: Dirichlet's L-function (of a nontrivial character) at 1 is nonzero – this uses analysis

Thank you for your attention!

I hope that was of some help.