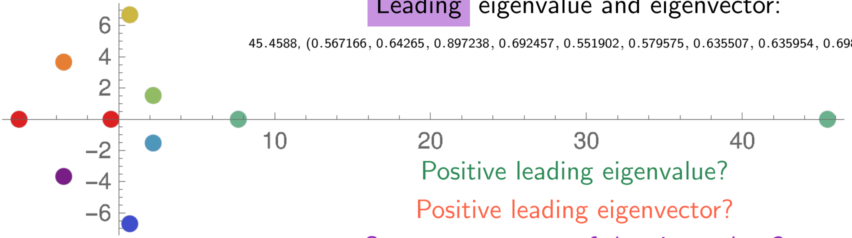
What is...the Perron–Frobenius theorem?

Or: The leading terms.

A motivating example

$$\begin{pmatrix} 3 & 0 & 5 & 1 & 8 & 7 & 0 & 1 & 4 & 7 \\ 4 & 8 & 0 & 6 & 3 & 4 & 2 & 6 & 8 & 3 \\ 8 & 6 & 6 & 7 & 6 & 0 & 9 & 4 & 8 & 5 \\ 3 & 7 & 7 & 1 & 5 & 6 & 4 & 1 & 7 & 4 \\ 4 & 0 & 3 & 4 & 4 & 8 & 8 & 1 & 4 & 2 \\ 0 & 3 & 7 & 3 & 2 & 4 & 2 & 2 & 3 & 8 \\ 6 & 3 & 6 & 1 & 5 & 6 & 1 & 6 & 4 & 4 \\ 2 & 4 & 0 & 2 & 8 & 8 & 1 & 4 & 8 & 6 \\ 6 & 7 & 6 & 3 & 4 & 2 & 9 & 6 & 5 & 0 \\ 0 & 6 & 9 & 9 & 8 & 3 & 9 & 9 & 1 & 9 \end{pmatrix}$$

What on earth is going on? Strange patterns with the eigenvalues and vectors:



Leading eigenvalue and eigenvector:

45.4588, (0.567166, 0.64265, 0.897238, 0.692457, 0.551902, 0.579575, 0.635507, 0.635954, 0.698596, 1)

Positive leading eigenvalue?

Positive leading eigenvector?

Strange symmetry of the eigenvalues?

Negative patterns?

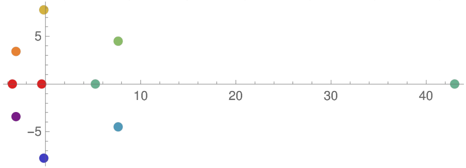
Non-negative. The pattern persist:

$$\begin{pmatrix} 1 & 3 & 8 & 6 & 6 & 2 & 3 & 6 & 8 & 7 \\ 5 & 5 & 1 & 8 & 3 & 0 & 3 & 3 & 7 & 6 \\ 0 & 1 & 6 & 3 & 6 & 7 & 5 & 3 & 9 & 0 \\ 1 & 0 & 2 & 8 & 7 & 2 & 8 & 8 & 3 & 9 \\ 9 & 2 & 6 & 1 & 9 & 6 & 3 & 2 & 6 & 5 \\ 6 & 7 & 0 & 1 & 4 & 5 & 9 & 0 & 4 & 5 \\ 0 & 4 & 8 & 4 & 1 & 4 & 0 & 2 & 5 & 6 \\ 6 & 6 & 1 & 5 & 7 & 4 & 2 & 7 & 3 & 0 \\ 7 & 7 & 3 & 0 & 2 & 0 & 6 & 4 & 8 & 2 \\ 6 & 7 & 9 & 3 & 1 & 2 & 1 & 8 & 7 & 4 \end{pmatrix} \rightsquigarrow$$

Leading eigenvalue and eigenvector:

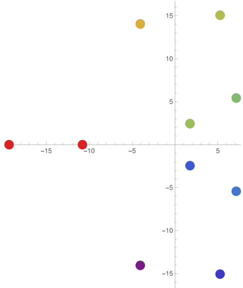
42.9948

(1.05024, 0.889464, 0.800641, 1.02602, 1.05438, 0.850121, 0.704574, 0.893187, 0.796941, 1)

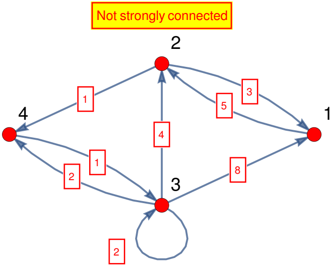


Negative. The pattern breaks:

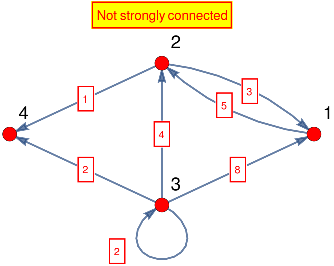
$$\begin{pmatrix} -4 & 0 & 1 & -2 & 0 & -5 & 8 & 6 & 8 & 3 \\ -9 & -9 & 7 & 5 & 6 & 8 & -6 & 5 & 1 & 1 \\ 8 & 3 & -4 & -3 & -9 & 4 & -8 & -8 & -6 & 7 \\ 0 & -4 & -4 & -4 & -4 & 5 & 3 & -4 & 5 & -7 \\ 0 & 3 & -2 & 2 & 5 & 1 & -2 & 0 & 9 & 8 \\ 6 & 8 & 0 & -6 & -7 & 3 & -7 & -9 & -4 & -4 \\ -8 & 8 & 5 & 6 & -1 & 3 & 0 & -3 & -3 & 0 \\ 4 & 3 & -1 & -9 & 6 & -4 & 2 & -3 & -1 & 7 \\ -2 & 6 & 2 & -6 & -8 & -4 & -5 & 0 & 2 & -1 \\ -6 & -1 & -1 & 5 & -7 & 7 & 4 & 4 & 9 & 4 \end{pmatrix} \rightsquigarrow$$



Let us talk about graphs



$$\leftrightsquigarrow \begin{pmatrix} 0 & \boxed{3} & \boxed{8} & 0 \\ \boxed{5} & 0 & \boxed{4} & 0 \\ 0 & 0 & \boxed{2} & \boxed{1} \\ 0 & \boxed{1} & \boxed{2} & 0 \end{pmatrix}$$



$$\leftrightsquigarrow \begin{pmatrix} 0 & \boxed{3} & \boxed{8} & 0 \\ \boxed{5} & 0 & \boxed{4} & 0 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & \boxed{1} & \boxed{2} & 0 \end{pmatrix}$$

A matrix valued in $\mathbb{N}_0$ is called irreducible if its graph is strongly connected.

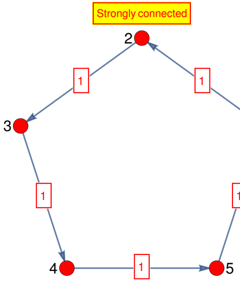
Enter, the theorem!

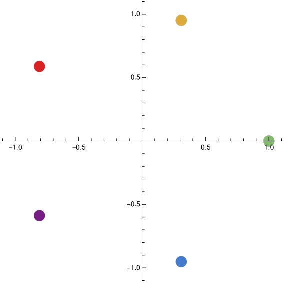
Let M be an irreducible matrix with entries in $\mathbb{N}_0$. Then:

(a) There exists a unique eigenvalue $pf \in \mathbb{R}_{>0}$ of M whose absolute value is bigger than those of other eigenvalues The leading eigenvalue

(b) Up to scalars, there is a unique eigenvector PF with entries from $\mathbb{R}_{>0}$, and it has eigenvalue pf The leading eigenvector

(c) The only eigenvectors with the same absolute value as pf are on the same circle as pf Symmetry of the eigenvalues





Question. Will a population become extinct?

Model. x_i^j is the number of members of the ith age group at the jth snapshot in time. $M = (m_{ij})$ transition matrix between the snapshots.

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & \dots & m_{1N} \\ m_{21} & 0 & 0 & 0 & 0 & \dots \\ 0 & m_{32} & 0 & 0 & 0 & \ddots \\ 0 & 0 & m_{43} & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1^j \\ x_2^j \\ x_3^j \\ x_4^j \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1^{j+1} \\ x_2^{j+1} \\ x_3^{j+1} \\ x_4^{j+1} \\ \vdots \end{pmatrix}$$

First row: Contribution of each age group to the reproduction
Lower diagonal: Transition from age group i to $i+1$

(a) If $pf(M) > 1$, then the population will grow without limit

(b) If $pf(M) < 1$, then the population will become extinct

(c) If $pf(M) = 1$, then it depends

Thank you for your attention!

I hope that was of some help.