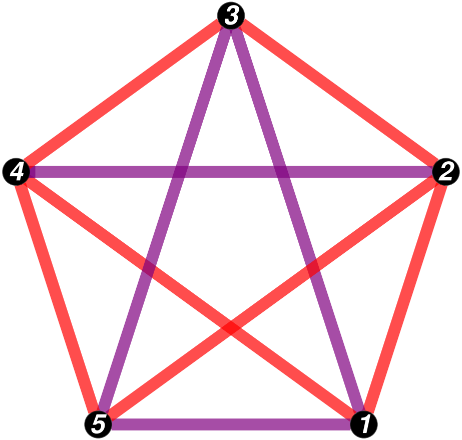
What is...coloring of numbers?

Or: From colors to set theory

Schur's masterpiece

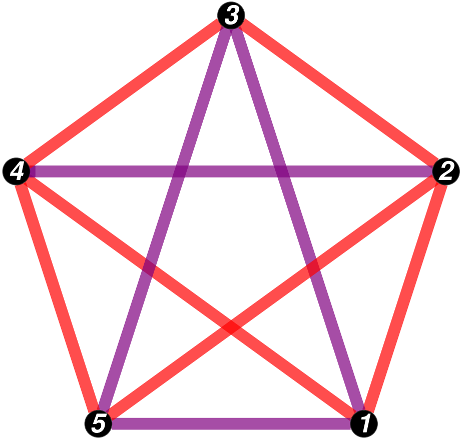
$\{1, 2, 3, 4, 5\}$:



- For any n there $\exists S(n) \leq n!e$ such that any n-coloring of $[S(n)] = \{1, \ldots, S(n)\}$ contains a monochromatic solution to $a + b = c$
- Schur's theorem was a starting point of many coloring problems à la Ramsey theory

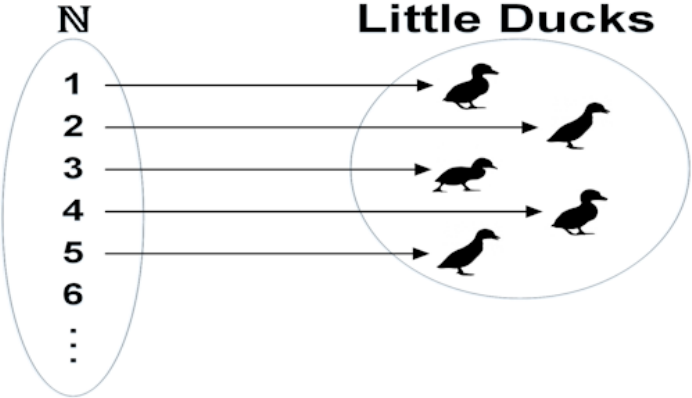
Proving Schur's masterpiece

$\{1, 2, 3, 4, 5\}$:



- Color the edge $i \leftrightarrow j$ of K_m for $m \approx n!e$ by the color of $i - j$
- Easy We find a triangle whose edges are colored in the same color
- The triangle is our solution

Going to bigger sets



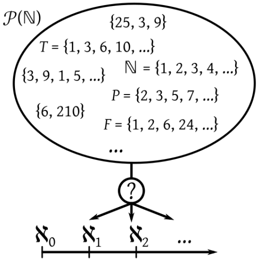
- Question What about colorings of $\mathbb{N}$ instead of finite sets?
- Question What about colorings of $\mathbb{R}$ instead of $\mathbb{N}$?
- We will use the variant of Schur's masterpiece searching for a monochromatic solution $a + b = c + d$

Enter, the theorems

We have:

- Schur's masterpiece works for any finite coloring of $\mathbb{N}$
- Schur's masterpiece works for any countable coloring of $\mathbb{R}$ if CH is false
- Schur's masterpiece fails for some countable coloring of $\mathbb{R}$ if CH is true

- CH = continuum hypothesis = there is no set whose cardinality is strictly between that of $\mathbb{N}$ and that of $\mathbb{R}$; this is independent of usual set theory



- The above is thus a combinatorial statement independent of usual set theory

An interesting boundary case

- Schur's masterpiece works for any finite coloring of $\mathbb{R}$
- Schur's masterpiece works for any countable coloring of V for V a $\mathbb{Q}$-vector space with $\dim_{\mathbb{Q}} V > \dim_{\mathbb{Q}} \mathbb{R}$

Thank you for your attention!

I hope that was of some help.