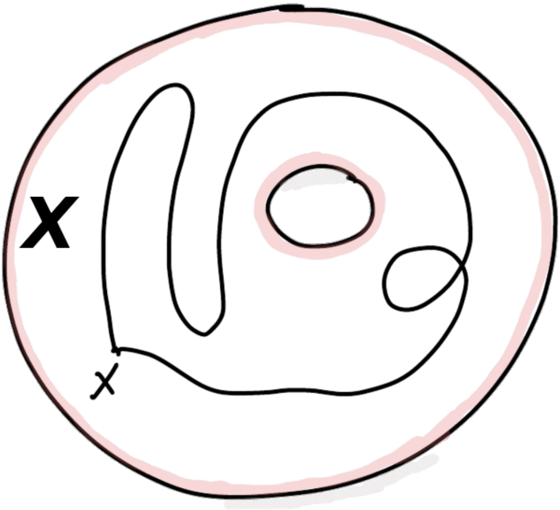
What is...homotopy of spheres?

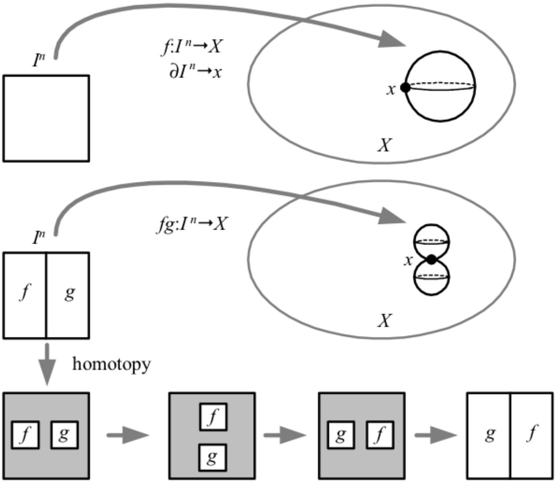
Or: Surprisingly hard!?

The fundamental group π_1



- π_1 measures how loops arrange in spaces
- π_1 is a great low dim invariant and fairly computable

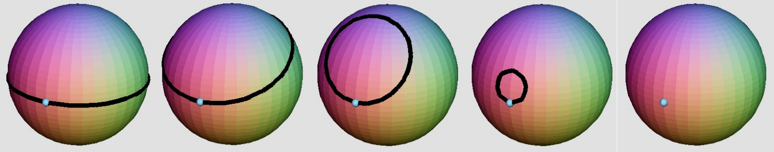
The higher ones π_n



- π_n measures how *n* spheres arrange in spaces
- π_n is a great *n* dim invariant: a bunch of numbers associated to a space
- What about computability?

What about the sphere?

$\pi_1\Bigg($  $\Bigg) \cong 1$



- $\pi_1(S^2)$ is easy to compute
- $\pi_n(S^2)$ = of the form $\mathbb{Z}^{\oplus n_0} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_l\mathbb{Z}$ = a bunch of numbers $n_0, \dots, n_l$
- Can we say anything about $\pi_n(S^2)$?

Enter, the theorems

We know infinitely many entries of the homology table $\pi_n(S^k)$:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2\times\mathbb{Z}/12$	$(\mathbb{Z}/2)^2\times\mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2\times\mathbb{Z}/12$	$(\mathbb{Z}/2)^2\times\mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\times\mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3\times\mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2\times\mathbb{Z}/12\times\mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2\times\mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

- But honestly, we know essentially nothing (only the "easy" bits are known)
- the S^0+S^1 rows are completely known, but already the S^2 row is widely open
- The colored bits are known for all n, k A slightly fattened diagonal
- The white bits are widely open in general

The north east is difficult to compute!

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3 \times \mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

- $\pi_3(S^2) \cong \mathbb{Z}$ is very hard to imagine Hopf fibration

- This kind of indicates that this is supposed to be hard
- Higher homotopy groups are algorithmically computable but the problem is still very hard ($W[1]$-hard with respect to n)

Thank you for your attention!

I hope that was of some help.