What are...expander graphs?

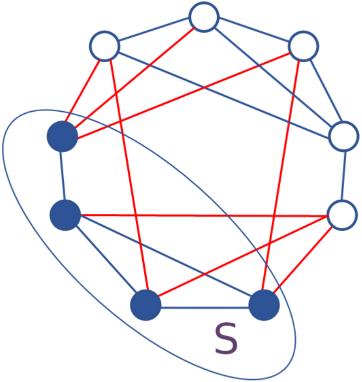
Or: Sparse and connected

Cutting cakes, ah sorry, graphs

Easy to cut:

- Problem We want graphs that are hard to cut yet have few edges
- Such graphs are called expanders
- It is not clear why such graphs exist

Measuring "Hard to cut"

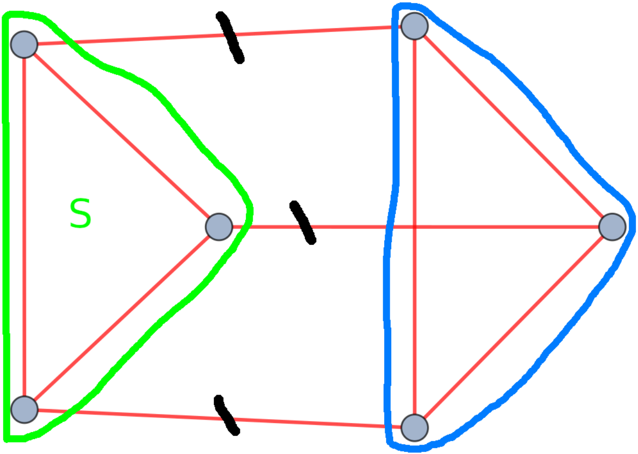


, $|\partial S| = 8$

- Given a subset S of the vertices of G, let ∂S be the boundary of S
- Let $h(G) = \min_{S, 0<|S|\leq n/2} |\partial S|/|S|$ Edge expansion or Cheeger's constant
- Slogan Large $h(G)$ means it is hard to cut G, small $h(G)$ means bottleneck
- Goal Find G with few edges and large $h(G)$

Bottlenecks and friends

$G_3 =$



- Take two complete graphs K_n; above $k = 3$
- Connected i vertices one-by-one and get graphs $G_0, G_1, G_2, \ldots, G_n$
- Then $h(G_0) = 0$, $h(G_1) = 1/n$, $h(G_2) = 2/n$, ..., $h(G_n) = n/n = 1$

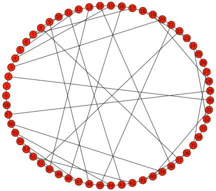
Enter, the theorem

Families of expanders exist

- Here is a definition:

> **Definition.** A sequence of (non-oriented, finite) graphs $(\Gamma_n)_{n\geq 1}$ is a *family of expanding graphs* if
>
> - The number of vertices of Γ_n tends to infinity as n tends to infinity;
> - There exists $k \geq 1$ such that the degree of *each* vertex of *each* graph is at most k (the graphs are not too dense);
> - There exists $\delta > 0$ such that $h(\Gamma_n) \geq \delta$ for all n (the Cheeger constant is uniformly bounded away from zero).
>
> Such graphs are simultaneously sparse and highly connected.

- Example Vertices $\{0, ..., p(prime) - 1\}$, connect $a \neq 0$ to $a \pm 1$ mod p and a^{-1} mod p, and 0 to $0, 1, p-1$, gives a family of expanders



Computer networks, brains, tramways, more...

- ▶ The above should have high $h(G)$ to avoid bottlenecks
- ▶ The above should have few connections increase efficiency
- ▶ There you go: expanders show up to save the day

Thank you for your attention!

I hope that was of some help.