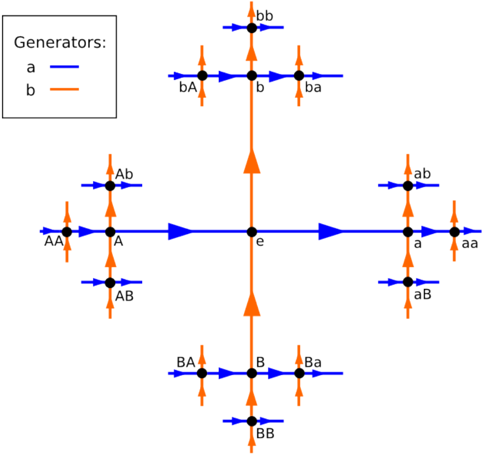
What is...the Novikov–Boone–Britton theorem?

Or: Sometimes life is not decidable

Words for free groups

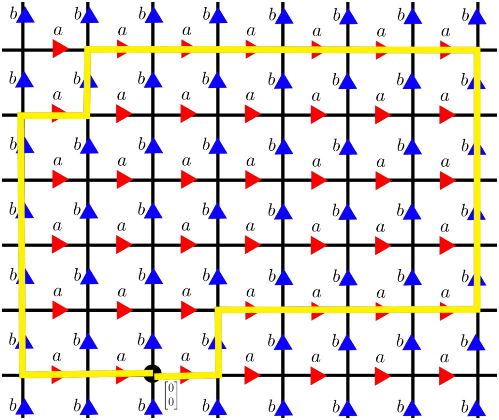
$F_2 \rightsquigarrow$



$, \quad A = a^{-1}, B = b^{-1}$

- Fix a finite alphabet $S = \{a, b, \ldots\} \rightsquigarrow$ free group on S
- A group word in S is a finite concatenation of symbols from $S \cup S^{-1} = \{a^{\pm 1}, b^{\pm 1}, \ldots\}$
- Example For $S = \{a, b\}$ some words are illustrated above

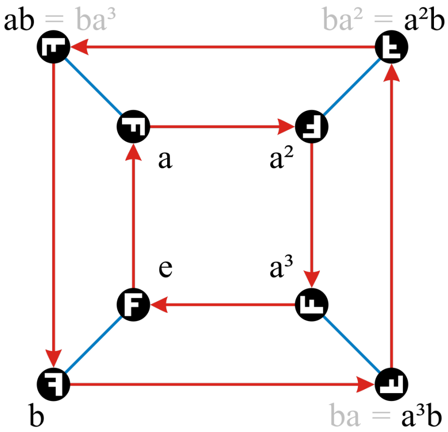
Words for groups

$\mathbb{Z}^2 \leftrightsquigarrow$ 

$, \quad ab = ba$

- Every group is a quotient of a free group
- We thus get the notion of words for a group
- Difference to the free groups: we now might have $w = w'$

The word problem

$D_8 \leftrightsquigarrow$ 

, $ab = ba^3$

- Word problem Can we decide whether two words represent the same element?
- Example Above we have $ab = baaa$

Enter, the theorem

Finitely presented groups with undecidable word problem exist

- Here is an explicit example

$$
\begin{array}{llll}
\langle & a,b,c,d,e,p,q,r,t,k & | & \\
& p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\
& p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\
& p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\
& p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\
& p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\
& aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\
& bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\
& cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\
& dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\
& eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 & \rangle
\end{array}
$$

- Undecidable roughly means that you cannot find an algorithm to check whether $w = w'$

Most groups = dragons

- It is quite difficult to explicitly find a group with undecidable word problem
- However, “most” groups should have an undecidable word problem
- As very often, we are biased towards “easy” groups and forget about the dragons

Thank you for your attention!

I hope that was of some help.