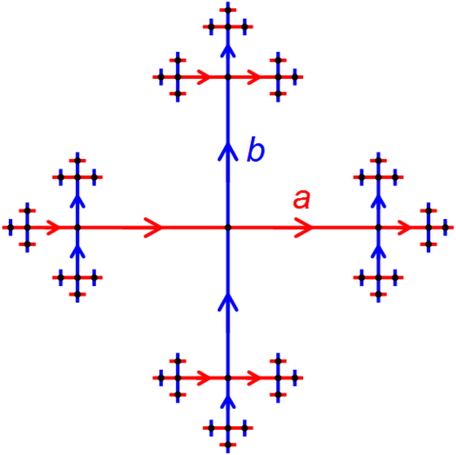
What is...Bass–Serre theory?

Or: Trees and groups

Studying groups by their actions

$F_2 = \langle a, b \rangle$ acts on

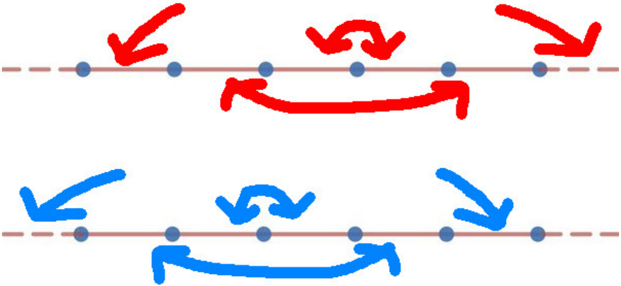


- Geometric group theory = study groups via their actions on geometric spaces
- Example The free group F_2 acts on the above tree, which in turn is a geometric spaces
- This video The "1d" part of the theory: groups acting on trees

Groups from their actions – not quite

$\mathbb{Z} = \langle a \rangle$ acts on

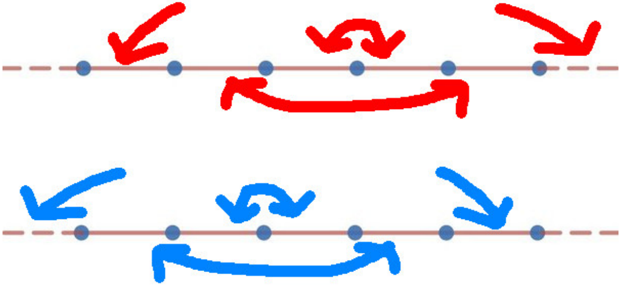
$D_\infty =$
$\langle a, b | a^2 = b^2 = 1 \rangle$ acts on

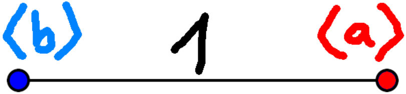


- $\mathbb{Z}$ acts on a line by translation; the quotient is a circle S^1
- ∞dihedral group acts on a line by reflection; the quotient is an interval $[0, 1]$
- We have $\pi_1(S^1) \cong \mathbb{Z}$ great! but $\pi_1([0, 1]) \not\cong D_\infty$ bad!

Groups from their actions

$D_\infty = \langle a, b | a^2 = b^2 = 1 \rangle$ acts on





observe: $D_\infty \cong \mathbb{Z}/2\mathbb{Z} *_1 \mathbb{Z}/2\mathbb{Z}$

- Bass–Serre: Keep a bit more data in the quotient X
- For example, keep the stabilizers
- Then it should hold that $\pi(X, v) \cong G$

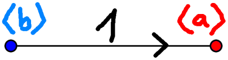
Enter, the theorem

Let G be a group acting on a tree T without inversions
Let X be the quotient graph of groups and let v be a base-vertex

$$\pi_1(X, v) \cong G$$

We recovered G from its action on T

- Graph of groups $\approx$ graphs whose vertices and edges are decorated by groups



This is done in a certain way; roughly, for $e \colon v \to w$:

- $G(e)$ is a distinguished subgroup of $G(v)$
- $G(e)$ embeds into $G(w)$ with a fixed embedding t_e

- There is an associated notion of fundamental group of these beasts

One key upshot

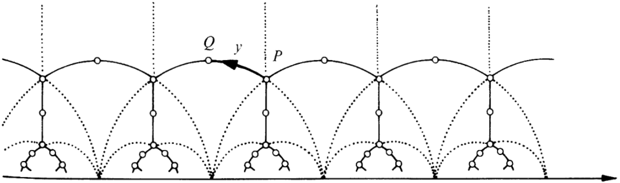
(c) $G = \mathbf{SL}_2(\mathbf{Z})$

This group acts in a well-known way on the half-plane $H = \{z | \mathrm{Im}(z) > 0\}$. Let y be the circular arc consisting of the points $z = e^{i\theta}$ with $\pi/3 \leqslant \theta \leqslant \pi/2$; its origin is the point $P = e^{\pi i/3}$ and its terminus is the point $Q = i$. Let X be the union of the transforms of y by G. One can show that X is a *tree* (or rather, the geometric realization of a tree) on which G acts with the segment PQ as fundamental domain. We have

$$G_P = \mathbf{Z}/6\mathbf{Z}, \qquad G_Q = \mathbf{Z}/4\mathbf{Z}, \qquad G_y = \mathbf{Z}/2\mathbf{Z},$$

so we recover the classical isomorphism between $\mathbf{SL}_2(\mathbf{Z})$ and

$$(\mathbf{Z}/4\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} (\mathbf{Z}/6\mathbf{Z}).$$



- The fundamental theorem of Bass–Serre theory $\Rightarrow$ a presentation of G
- Generators $G(v)$ and the t_e
- Relations The ones for $G(v)$ plus “connectivity relations”

Thank you for your attention!

I hope that was of some help.