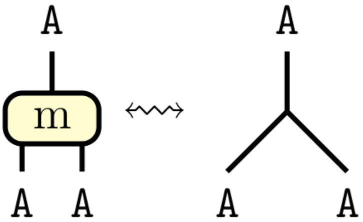
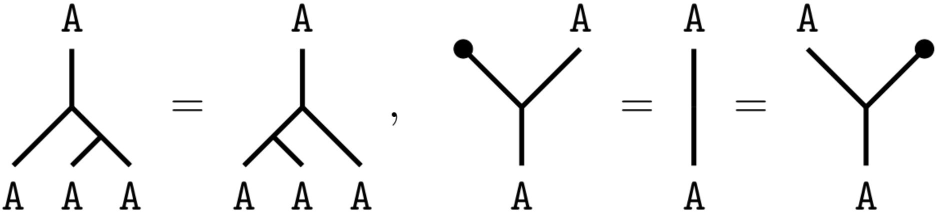
What is...a Hopf algebra?

Or: Why we comultiply

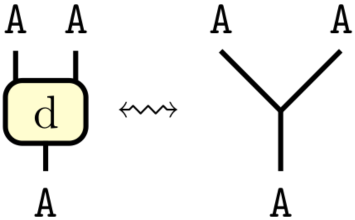
Multiplication is asymmetric

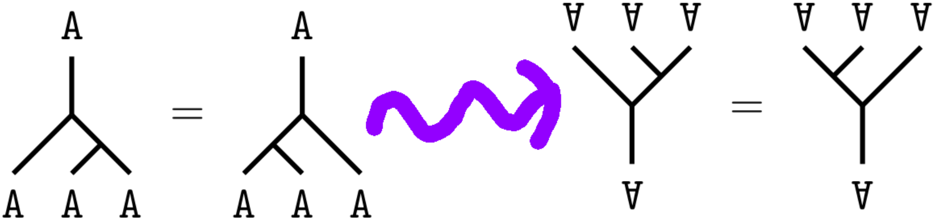




- Multiplication m is everywhere, e.g. $\mathbb{C}[G]$=group algebra, then $\mathrm{m}(g,h)=gh$ works
- We can illustrate this using a trivalent vertex
- Doing this shows asymmetry: why not flip the picture?

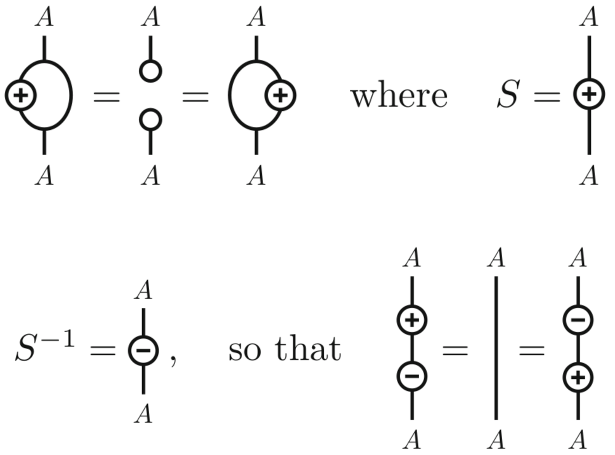
Flip pictures





- Comultiplication d is defined by flipping the multiplication
- We assume the same axioms but flipped
- Example $\mathbb{C}[G]$=group algebra, then $\mathrm{d}(g) = g \otimes g$ works

The antipode is a bit obscure

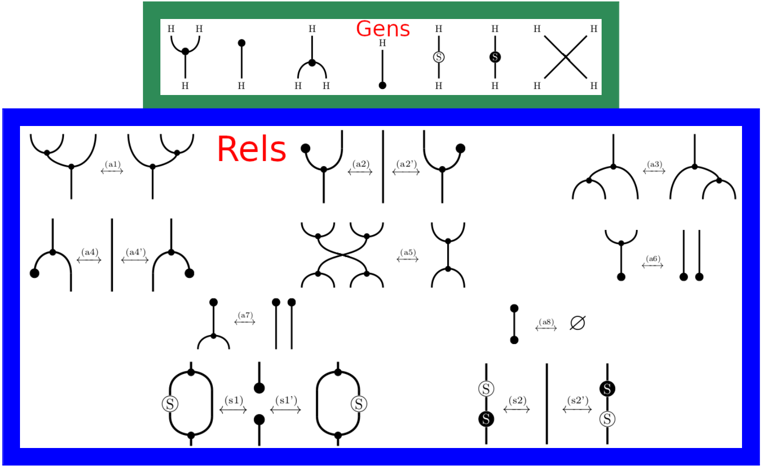


- Analyzing examples, as e.g. on the final slide, gives the antipode $\mathrm{S}\colon A \to A$
- Looks slightly obscure, but is actually good believe me ;-)
- Example $\mathbb{C}[G]$=group algebra, then $\mathrm{S}(g) = g^{-1}$ works

Enter, the theorem

Hopf algebras exist and are everywhere

- Here is the diagrammatic definition of a Hopf algebra:



- Example $\mathbb{C}[G]$=group algebra with structures as before

Examples, examples, examples

	Depending on	Comultiplication	Counit	Antipode	Commutative	Cocommutative	Remarks
group algebra KG	group G	$\Delta(g) = g \otimes g$ for all g in G	$\varepsilon(g) = 1$ for all g in G	$S(g) = g^{-1}$ for all g in G	if and only if G is abelian	yes	
functions f from a finite[14] group to K, K^G (with pointwise addition and multiplication)	finite group G	$\Delta(f)(x,y) = f(xy)$	$\varepsilon(f) = f(1_G)$	$S(f)(x) = f(x^{-1})$	yes	if and only if G is abelian	
Representative functions on a compact group	compact group G	$\Delta(f)(x,y) = f(xy)$	$\varepsilon(f) = f(1_G)$	$S(f)(x) = f(x^{-1})$	yes	if and only if G is abelian	Conversely, every commutative involutive reduced Hopf algebra over **C** with a finite Haar integral arises in this way, giving one formulation of Tannaka–Krein duality.[15]
Regular functions on an algebraic group		$\Delta(f)(x,y) = f(xy)$	$\varepsilon(f) = f(1_G)$	$S(f)(x) = f(x^{-1})$	yes	if and only if G is abelian	Conversely, every commutative Hopf algebra over a field arises from a group scheme in this way, giving an antiequivalence of categories.[16]
Tensor algebra T(V)	vector space V	$\Delta(x) = x \otimes 1 + 1 \otimes x$, x in V, $\Delta(1) = 1 \otimes 1$	$\varepsilon(x) = 0$	$S(x) = -x$ for all x in $T^1(V)$ (and extended to higher tensor powers)	If and only if dim(V)=0,1	yes	symmetric algebra and exterior algebra (which are quotients of the tensor algebra) are also Hopf algebras with this definition of the comultiplication, counit and antipode

There are trillions of examples – too many to fit on this slide

Thank you for your attention!

I hope that was of some help.