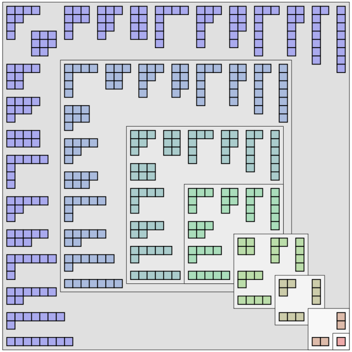
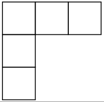
What is...the partition number?

Or: To count or not to count...

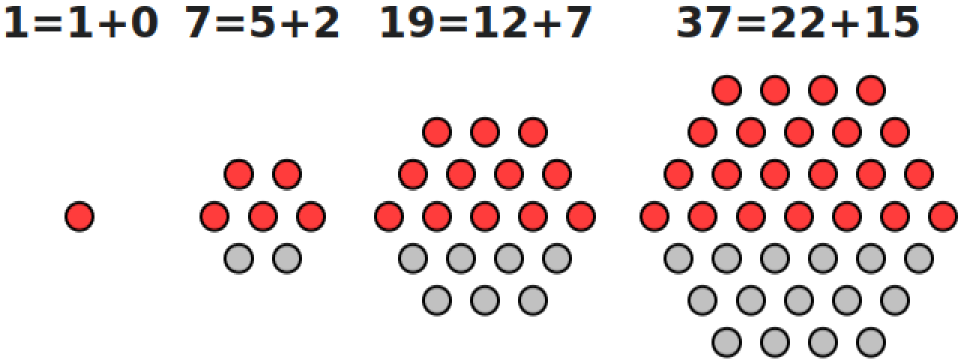
(Integer) partitions



$3+1+1 \leftrightsquigarrow$ 

- Partitions of n = a way of writing n as a sum of positive integers, ignoring order
- Counting them is a classical problem: "find $p(n)$ = number of partitions"
- But how do we do this?

Partitions and pentagons



$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots \quad , \quad p(n) = \sum_{k \neq 0} (-1)^{k-1} p(n - g_k)$$

- $p(n)$ can be computed recursively using pentagonal numbers g_k
- Problem The formula is recursive, so is not really counting

Partitions and generating functions

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \sum_{i=0}^{\infty} q^{ji} = \prod_{j=1}^{\infty} (1-q^j)^{-1}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)x^n &= \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k}\right) \\ &= \left(1+x+x^2+x^3+\cdots\right)\left(1+x^2+x^4+x^6+\cdots\right)\left(1+x^3+x^6+x^9+\cdots\right)\cdots \\ &= \frac{1}{1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\cdots} \\ &= 1 \Big/ \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.\end{aligned}$$

Example:

$[1,0,1,0,1,0...] \quad \leftrightarrow \quad 1+x^2+x^4+x^6.... \quad = \frac{1}{1-x^2}$

Multiplying the generating function by 2 gives

$\frac{2}{1-x^2} = \; 2+2x^2+2x^4+2x^6..$

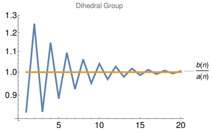
- $p(n)$ can be computed using a generating function (via Taylor expansion)
- Problem The formula is still uses a calculation, so is not really counting

Enter, the theorem

Asymptotically:

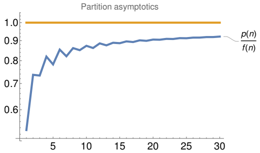
$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

- Asymptotic means $\lim_{n\to\infty} b(n)/a(n) \to 1$:

$a(n) \sim b(n)$: 

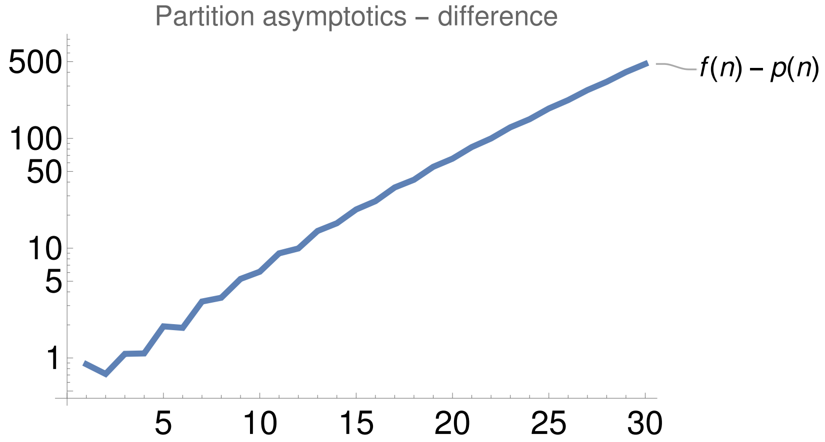
- Here is a comparison between $p(n)$ and its asymptotic formula:

$$p(n) \sim f(n) = \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right):$$



- This is somewhat still not really counting

Its really not counting



- The asymptotic $p(n) \sim f(n)$ is good
- The formula $f(n) = \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$ is good
- This is still not really counting: the difference can get arbitrary large

Thank you for your attention!

I hope that was of some help.