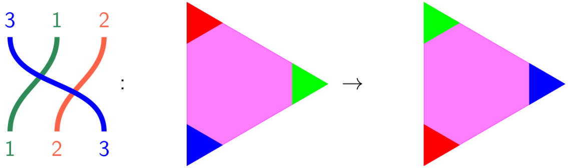
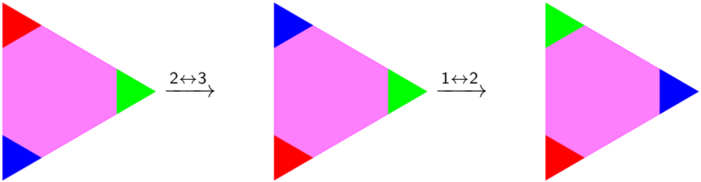
What are...Coxeter complexes?

Or: Spheres and points

Acting on triangles

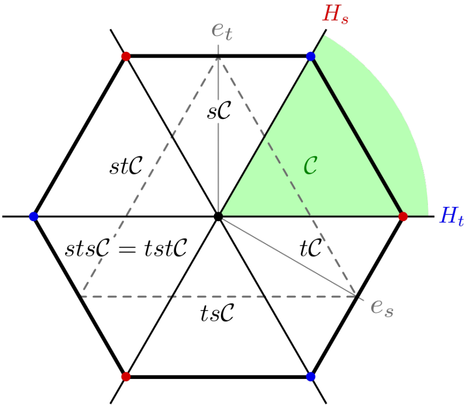
The symmetric group in three letters acts on a triangle via the rule
"green=1, red=2, blue=3, and then permute":





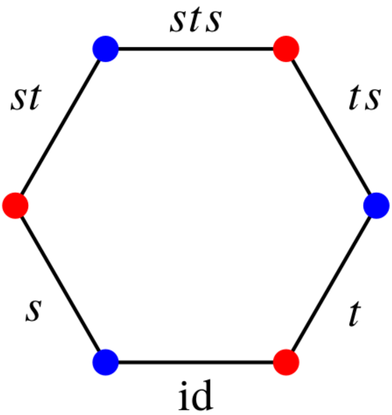
- **Task** Associate a geometric object to the symmetric group S_n
- **Starting point** S_n acts on an $n-1$ simplex (triangle for $n=3$)
- The action is generated by the **reflections** for $(i, i+1) \in S_n$

Reflections in triangles



- $s = (1, 2)$ and $t = (2, 3)$ generate the S_3-action on the triangle
- Take the reflection hyperplanes H_s and H_t for them and their orbits
- The hyperplane complement is separated into chambers where S_3 acts faithfully

Gluing pieces into a sphere

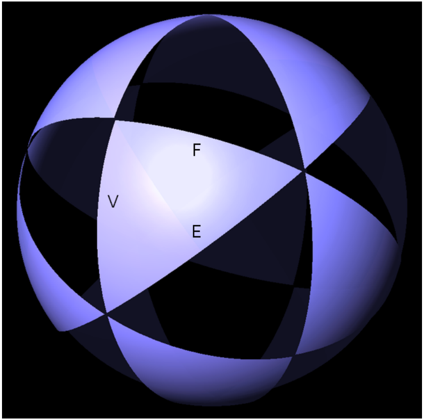


- $S_3 = \{id, s, t, st, ts, sts = tst\}$
- Mark one/any chamber *id* and follow the reflection action by s and t
- The polygon that is traced out is the Coxeter complex of S_3

Enter, the theorem

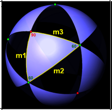
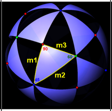
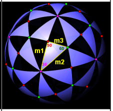
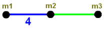
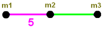
The S_n Coxeter complex can be defined for any n and is homeomorphic to a sphere S^{n-2}

$\cong$

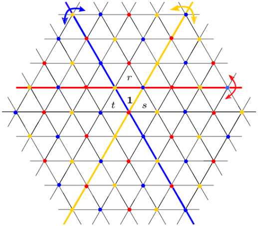


- Above The Coxeter complex of S_4
- Slide before The Coxeter complex for S_3 is homeomorphic to S^1

Its even more general

Coxeter group	A_3	BC_3	H_3
	[3,3]	[4,3]	[5,3]
Funda-mental domain			
Coxeter-Dynkin diagram			

$\tilde{A}_2$:



- The Coxeter complex $C(G)$ can be defined for any reflection group $G = (W, S)$
- Theorem $C(G) \cong S^{|S|-1}$ for G finite and $C(G)$ is contractible otherwise

Thank you for your attention!

I hope that was of some help.