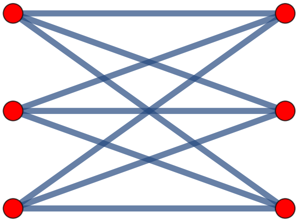
What is...the complexity of embeddings?

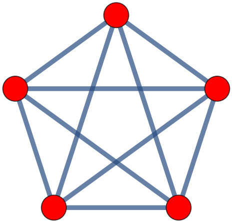
Or: How difficult is drawing?

Planar graphs

not planar: $K_{3,3} =$

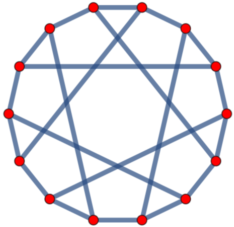


not planar: $K_5 =$

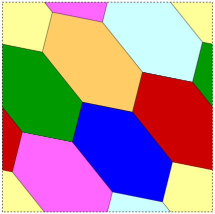


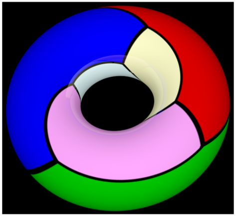
- **Planar graph** = a graph that can be drawn (without intersections) in the plane
- **Example** $K_{3,3}$ and K_5 are not planar, but that is a bit difficult to see
- **Question** How efficient can one check this (say with a machine)?

Graphs on surfaces



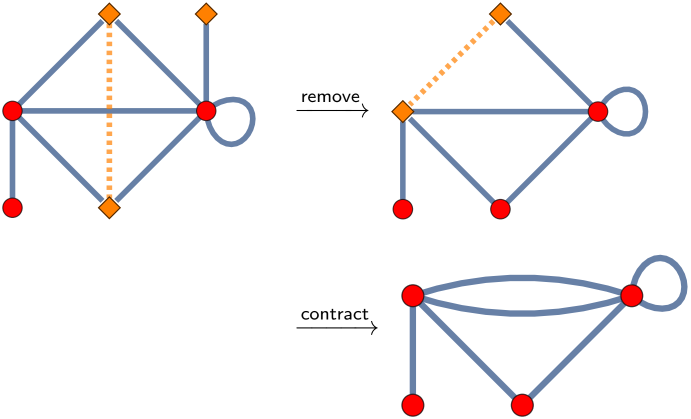
The Heawood graph on a torus:





- The Heawood graph, $K_{3,3}$ and K_5 can be drawn on torus (genus 1=1 hole)
- In fact, every graphs can be drawn on some surface
- Question How efficient can one check this? How can we efficiently find the minimal (genus) surface for a given graph?

The thing with minors works...kind of...



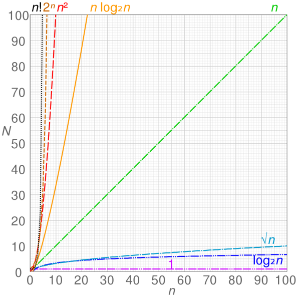
- G has H as a minor, if H is obtained from G via remove & contract
- A graph is planar $\Leftrightarrow$ it does not contain $K_{3,3}$ and K_5 as minors, and there is a similar statement for higher genus – we should exploit this, right?
- Problem For higher genus the list of forbidden minors gets insane (and is not known in general)

Enter, the theorem

For any fixed closed surface S $\exists c \in \mathbb{N}$ and a linear time algorithm that for an arbitrary given graph G either:

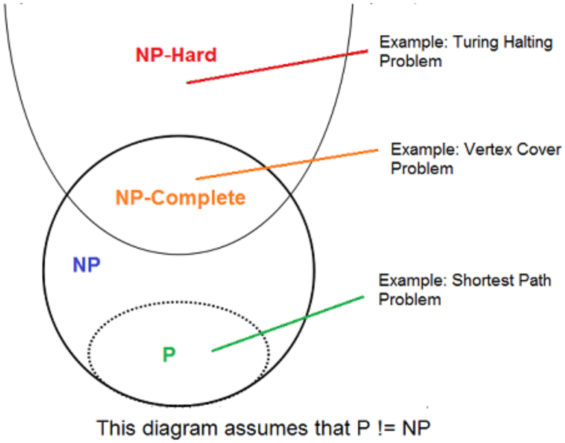
(i) Finds an embedding of G in S, or

(ii) Identifies a minimal forbidden subgraph for embeddability in S whose size is bounded by c



- More than existence: The algorithm constructs the embedding
- This theorem can also find the forbidden minors

Finding the genus is NP complete



- Previous slides: checking embeddability for a fixed surfaces is super easy
- However, finding the minimal surface for a fixed graph is NP-complete "=" difficult
- This somewhat walks along the border of "P *vs.* NP"

Thank you for your attention!

I hope that was of some help.