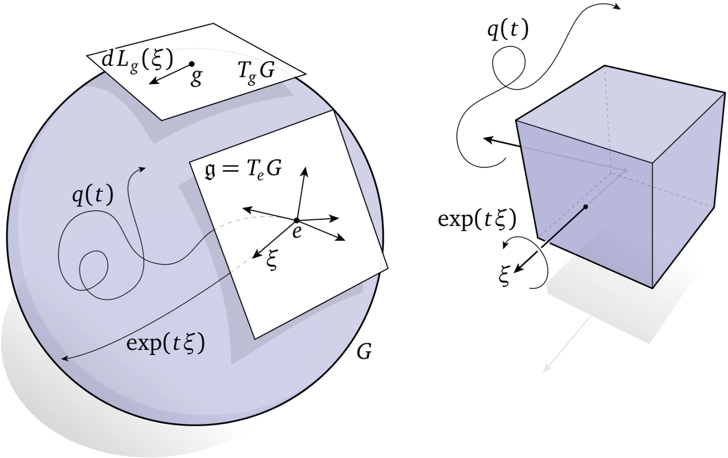
What is...Freudenthal's magic square?

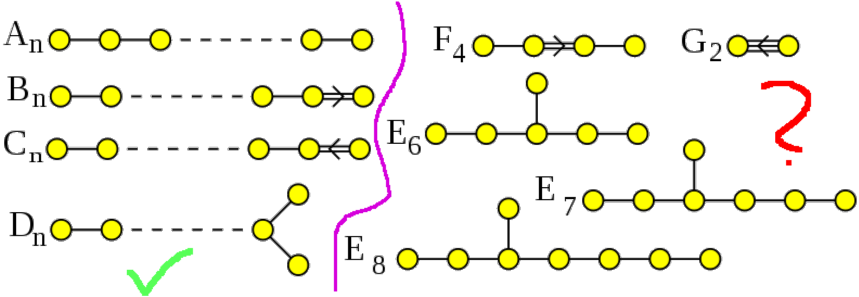
Or: Exceptional!

Lie groups and algebras



- Lie groups $\leftrightsquigarrow$ continuous Galois theory $\leftrightsquigarrow$ continuous symmetries; Lie algebras $\leftrightsquigarrow$ first order approximations of these
- Key formula for going between Lie groups and algebras: $\det e^A = e^{\mathrm{tr} A}$
- Example The Lie group $\mathrm{SL}_2(\mathbb{C})$ (det=1) has $\mathfrak{sl}_2(\mathbb{C})$ (tr=0) as its Lie algebra

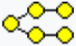
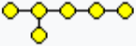
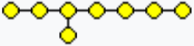
Classification is it!



$\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ $\leftrightsquigarrow$ A_n, B_n, C_n, D_n

- Question Can we classify the elements of Lie theory (simple Lie algebras)?
- Answer Yes we can (say over $\mathbb{C}$)!
  - We have the natural families $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$
  - A five "weird" exceptions
- Question Where do the weird exceptions come from?

One square to find them

A \ B	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1 	A_2 	C_3 	F_4 
$\mathbb{C}$	A_2 	$A_2 \times A_2$ 	A_5 	E_6 
$\mathbb{H}$	C_3 	A_5 	D_6 	E_7 
$\mathbb{O}$	F_4 	E_6 	E_7 	E_8 

- Freundenthal–Tits Construction of a Lie algebra / Dynkin diagram from a pair of division algebras A, B
- Last video The key division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (octonions)
- $\mathbb{O}$ then creates all expectational types (G_2 appears as automorphism)

Enter, the theorem

For a pair of division algebras A, B let

$$L = \left(\mathfrak{der}(A) \oplus \mathfrak{der}(J_3(B))\right) \oplus \left(A_0 \otimes J_3(B)_0\right)$$

and the corresponding Lie algebras are

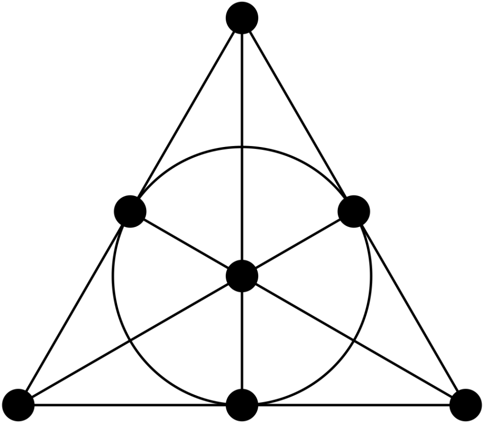
	B	**R**	**C**	**H**	**O**
A	***der*(A/B)**	**0**	**0**	$\mathfrak{sp}_1$	$\mathfrak{g}_2$
R	**0**	$\mathfrak{so}_3$	$\mathfrak{su}_3$	$\mathfrak{sp}_3$	$\mathfrak{f}_4$
C	**0**	$\mathfrak{su}_3$	$\mathfrak{su}_3 \oplus \mathfrak{su}_3$	$\mathfrak{su}_6$	$\mathfrak{e}_6$
H	$\mathfrak{sp}_1$	$\mathfrak{sp}_3$	$\mathfrak{su}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
O	$\mathfrak{g}_2$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

A \ B	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \times A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Thus, we get the exceptional types from $\mathbb{O}$

- $\mathfrak{der}$ = derivations; J = Jordan algebra
- $\mathfrak{su}_n$ is the compact version of $\mathfrak{sl}_n$

Fano again...?



- $\mathbb{O}$ can be constructed from the Fano plane, so exceptional Lie algebras can
- Many sporadic groups can be constructed from the Fano plane
- In some sense the Fano plane is thus the exceptional object in math

Thank you for your attention!

I hope that was of some help.