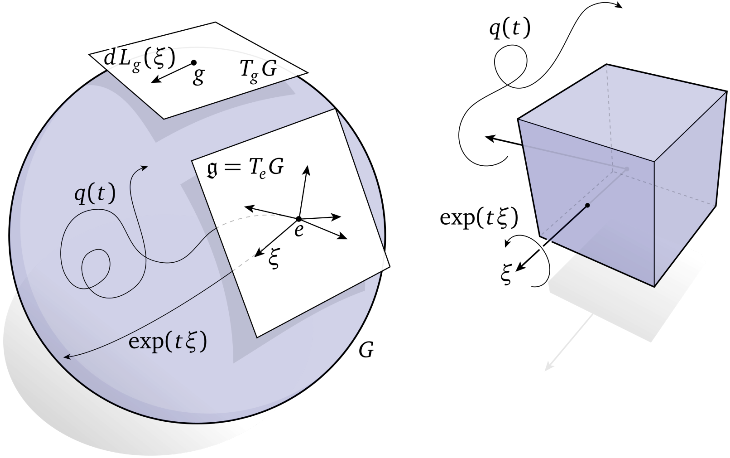
What are...crystal graphs?

Or: Low temperature behavior

Lie algebras and their representations



- Lie algebras (over $\mathbb{C}$) $\leftrightsquigarrow$ (first order approx. of) continuous symmetries
- Their representations $\leftrightsquigarrow$ vector spaces versions of continuous symmetries
- Task Find good models of (simple) Lie algebra representations

Generators and graphs

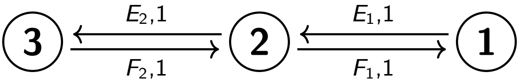
$\mathfrak{sl}_3(\mathbb{C}) = \{\text{complex 3-by-3 matrices with trace} = 0\}$ with generators :

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

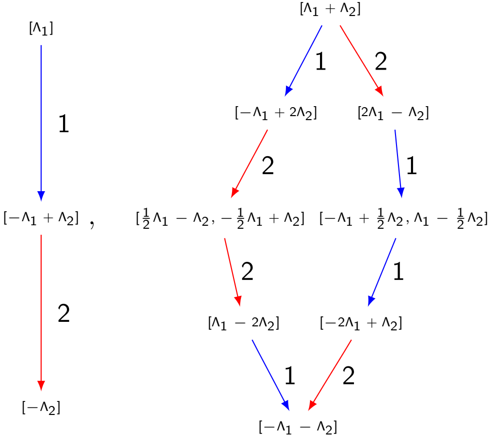
$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$V = \{e_1, e_2, e_3\}$ + matrix action:



- The Chevalley generators are "matrices" such as the ones above
- With these representations are labeled weighted graphs (above only E, Fs)
- Problem These graphs get messy fast and are not super helpful

Crystals



- Crystal = labeled graph; one label for every F_i
- Idea The crystal of a representation is the graph of leading terms of the action of F_i, e.g.

$$F_i \circlearrowright v_j = v_k + \text{friends} \Rightarrow \text{draw an edge } v_j \to v_k$$

Enter, the theorem

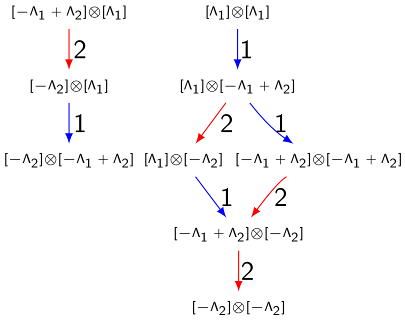
We have the following (many more nice things about crystals are true):

(i) Every representation has an associated crystal **Existence**

(ii) The crystal determines the representation **Uniqueness**

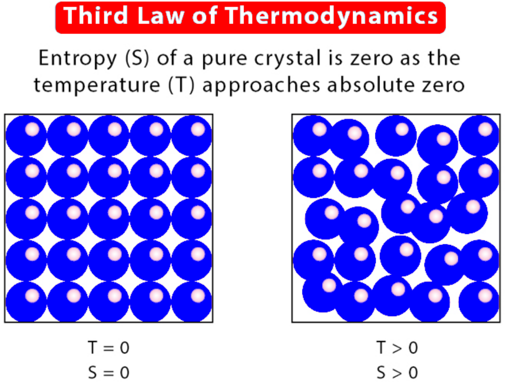
(iii) Connected components $\leftrightsquigarrow$ to simple representations **Combinatorial properties are encoded**

- Here is an example of a tensor product:



- Here connected components correspond to the tensor product decomposition

Temperature zero



- Big question How to define "leading term"?
- Trick There is a "quantum object" and a "canonical basis" where the action coefficients are in $\mathbb{Z}[q]$, e.g. $1 + 2q^2 + 2q^4 + q^6$
- Absolute zero Specializing $q = 0$ gives the leading term

Thank you for your attention!

I hope that was of some help.