What is...Hilbert's Satz 90?

Or: Number 90 is it!

Zahlbericht

Die Theorie

der

algebraischen Zahlkörper.

1897

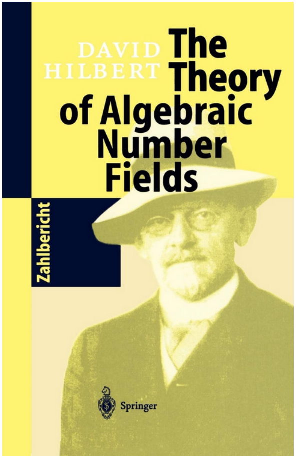
Bericht,

erstattet der Deutschen Mathematiker-Vereinigung

von

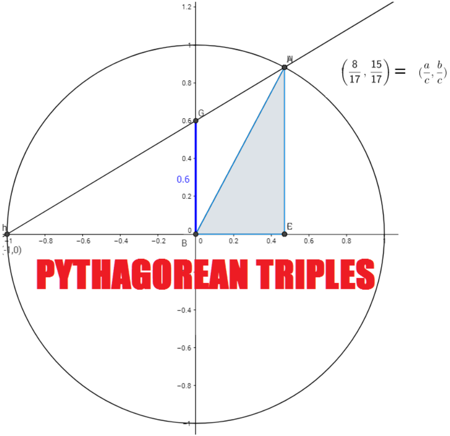
David Hilbert.

Satz 90. Jede ganze oder gebrochene Zahl A in K, deren Relativnorm in Bezug auf k gleich 1 ist, wird die symbolische $(1-S)$te Potenz einer gewissen ganzen Zahl B des Körpers K.



- In the famous Zahlbericht ($\approx$ report on numbers) Hilbert summarized algebraic number theory, and enriching and organizing the subject in ways that were to influence developments for decades
- The theorem we will see is number 90 in Hilbert's Zahlbericht

Pythagorean triples (PT)



- PT $=$ integer with $a^2 + b^2 = c^2$ $=$ rational points (x, y) on the unit circle
- Solution $\exists m, n \in \mathbb{Z}$ such that $(x, y) = (m^2 - n^2, 2mn)/(m^2 + n^2)$
- Question Where do the solutions come from?

Functional equations (FE)

Functional equations of the form	Possible Functions
$f(xy)=f(x)+f(y);\ x,y>0$	$f(x)=K\cdot\log_a x$
$f(xy)=f(x)\cdot f(y);\ x,y\in\mathbb{R}$	$f(x)=x^n$
$f(xy)=x\cdot f(y)+y\cdot f(x);\ x,y\in\mathbb{R}^+$	$f(x)=x\log x$
$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)};\ y\neq 0; f(y)\neq 0$	$f(x)=x^n$
$f(x+y)=f(x)+f(y)$	$f(x)=f(1)\cdot x$
$f(x+y)=f(x)\cdot f(y)$	$f(x)=\left[f(1)\right]^x$
$f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$	$f(x)=1\pm x^n$

- Functional equations are everywhere but often difficult
- The function equation

$$f(x)f(\zeta x)\dots f(\zeta^{n-1}x)=1 \text{ for } \zeta=\exp(2\pi i/n)$$

has nice solutions of the form

$$f(x)=\frac{g(x)}{g(\zeta x)} \text{ for } g\in\mathbb{C}(x)$$

- Question Where do the solutions come from?

Enter, the theorem

Consider the following

(i) E/F a finite cyclic field extension

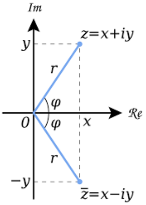
(ii) σ a generator of $Gal(E/F)$ of order n

(iii) $\alpha \in E$ of relative norm 1, i.e. $\alpha\sigma(\alpha)...\sigma^{n-1}(\alpha) = 1$

Then we can express α rationally:

$$\alpha = \beta/\sigma(\beta) \text{ for } \beta \in E$$

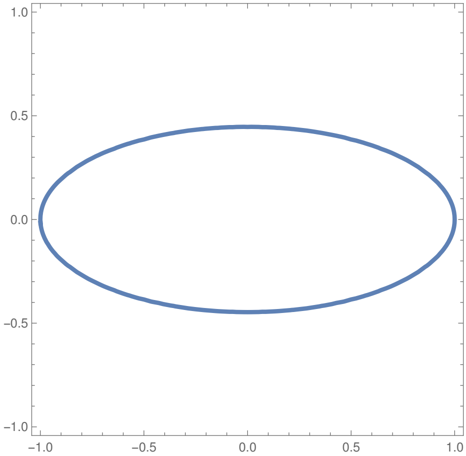
- PT example Take $\mathbb{Q}(i)/\mathbb{Q}$, σ = complex conjugation



- FE example Take $\mathbb{C}(x)/\mathbb{C}(x^n)$, $\sigma(x) = \zeta x$

Generalized PT

$x^2 + 5y^2 = 1$:



- One can also find solutions to the ellipse equation $x^2 + Dy^2 = 1$
- This follows by using $\mathbb{Q}(\sqrt{-D})/\mathbb{Q}$ in Hilbert's Satz 90
- The solutions are then of the form $(x, y) = (m^2 - Dn^2, 2mn)/(m^2 + Dn^2)$

Thank you for your attention!

I hope that was of some help.