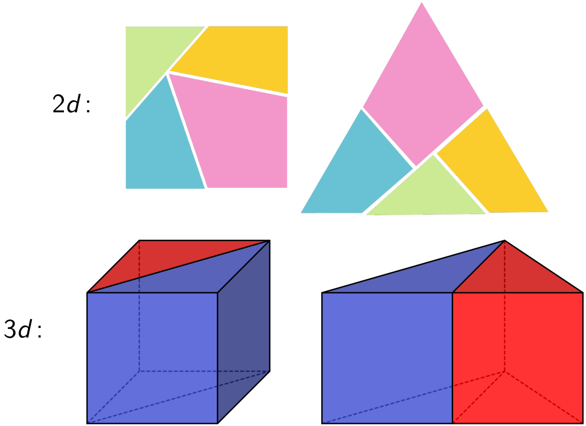
What are...Dehn invariants?

Or: Cutting polyhedra

Playing with scissors

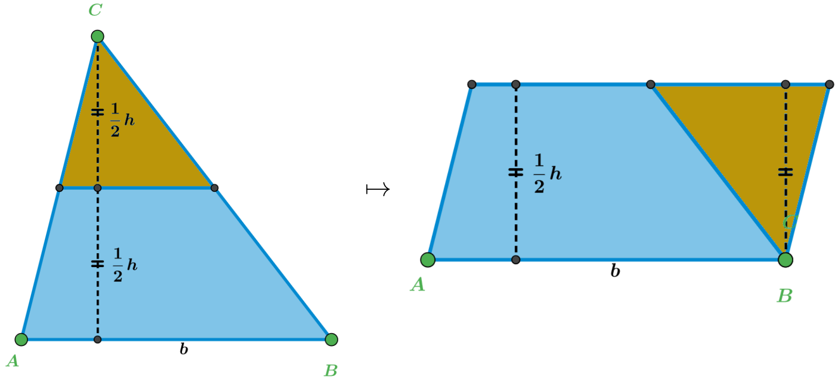


- Take two **polyhedra** of the same area/volume/4d volume/...
- **Question** Can we cut and reassemble one into the other?
- **Known since 1800ish** The answer is “Yes” in dimension 2

Hilbert's third problem

- In 1900 Hilbert gave very influential 23 problems for the 20th century
- One of them is the question from the previous slide (in fact, the original question was slightly different)
- Example Can one rearrange the cube into the other platonic solids?

No one likes calculus...!?



- In 2d geometry the formulas for the area of the basic polyhedral shapes have nice=geometry proofs
- In 3d geometry the formulas for the volumes of the basic polyhedral shapes have usually not so nice=analytic proofs
- Hilbert's question ⇝ Is it impossible to always have nice proofs?

Enter, the theorem

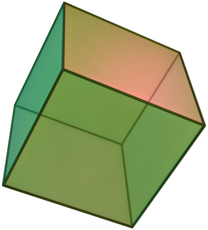
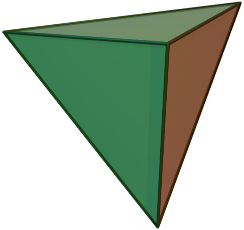
There exist invariants $D(P_i)$ for any polyhedron P_i such that:

(i) P_1, P_2 can be cut + reassemble into one another $\Rightarrow \big(D(P_1) = D(P_2)\big)$ Dehn

(ii) $\big(D(P_1) = D(P_2)\big) \Rightarrow P_1$, P_2 can be cut + reassemble into one another Sydler

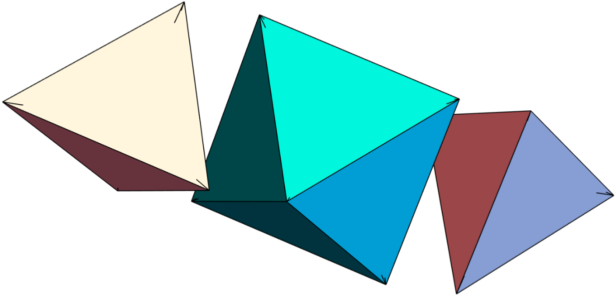
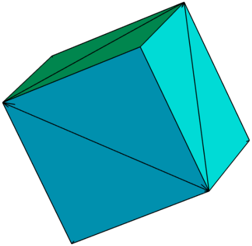
Here we assume that the polyhedra we consider have the same volume

- Example We have

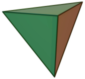
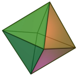
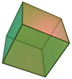
D () $= 0$ but D () $\neq 0$

- This implies that Hilbert was right (that is, it is not possible)

Two tetrahedron + octahedron = 2 cubes

 $\leftrightsquigarrow 2 \cdot$ 

- The Dehn–Sydler theorem is an if and only if theorem
- Example We have

$2 \cdot D\Big($  $\Big) + D\Big($  $\Big) = D\Big($  $\Big)$

and indeed one can build them from one another

Thank you for your attention!

I hope that was of some help.