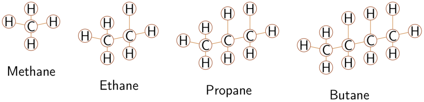
What is...tree counting?

Or: $n + 1$ and $n - 1$

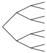
Trees

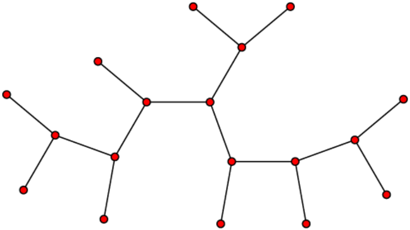
Examples

- Saturated hydrocarbons



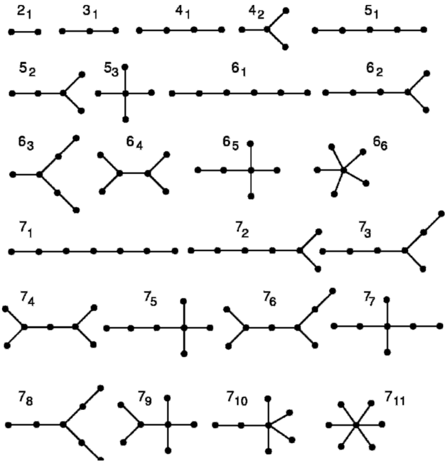
- A tournament tree





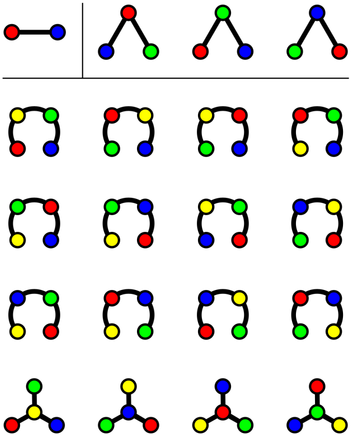
- Tree = a graph with out nontrivial cycles
- One can think of trees as easiest graphs and appear everywhere
- Task Count them!

Counting trees



- Turns out that counting trees is difficult
- This is meant in the sense that there no known closed formula
- Fun side fact #trees $\sim 0.5349496... \cdot n^{-5/2} \cdot 2.9557652...^n$

Counting colored trees

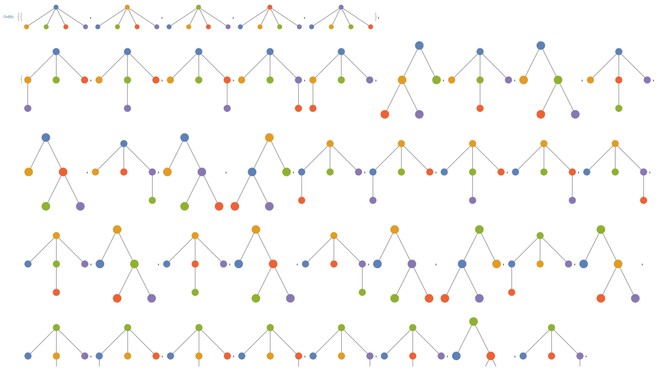


- Borchardt–Cayley ∼1860 Counting colored trees might be easier!
- The number sequence then is $1, 1, 3, 16, 125, \ldots$
- Turns out the answer gets easier when making the question more complicated

Enter, the theorem

The number of colored trees on $n+1$ vertices is

$$(n+1)^{n-1}$$

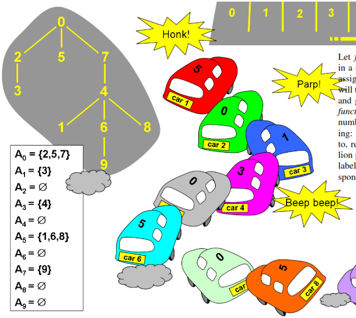


- Above things are shifted and usually people write n^{n-2}
- Many proofs are known and they are (brilliant but also) quite easy

More on $n+1$ and $n-1$

THEOREM OF THE DAY

The Parking Function Formula *The number of parking functions of order n is* $(n+1)^{n-1}$.



Let f be a mapping from cars $\{1,\dots,n\}$ to parking spaces $\{0,\dots,n-1\}$ in a one-way street. The cars arrive in order, with the i-th arrival being assigned position $f(i)$. If an earlier car already took that spot then car i will try the next place along until it parks or fails at the last parking space and gives up. If no car is forced to give up then f is called a *parking function*. In the picture here, only the first four cars get their intended number, but all cars at least manage to find a place, the allocation being: 2, 3, 5, 4, 7, 1, 6, 8, 9. So the function mapping 1, 2, 3, 4, 5, 6, 7, 8, 9 to, respectively, 5, 0, 1, 3, 0, 5, 0, 5, 7, is one of exactly one-hundred million parking functions for nine cars. By Cayley's Formula there are n^{n-2} labelled trees on n vertices; surely they may be put into one to one correspondence with the parking functions for $n-1$ cars?

Such a bijection is shown on the left (due to Jean Françon; this version is from a nice paper of Philippe Chassaing and Jean-François Marckert): the system of disjoint subsets, A_k, of $\{0,\dots,9\}$, bottom left, satisfies $\sum_{j=0}^{k} |A_j| > k$, for $k = 0,\dots,8$; this guarantees that the function mapping the elements of A_k to k is a parking function. The same condition guarantees that a labelled tree may be constructed breadth-first: build up the tree layer by layer, starting with vertex 0 at time 0; the vertex that is added at time t will have the elements of A_t added as children when the next layer is constructed.

Parking functions were introduced by Ronald Pyke in 1959 and independently by Alan Konheim and Benjamin Weiss in the context of data storage. They have come to be studied intensively as mathematical objects in their own right.

Web link: www.maths.qmul.ac.uk/~pjc/preprints/art.pdf is a nice glimpse of how professional mathematicians respond to stimulating ideas such as the above. And here is the result: www.combinatorics.org/ojs/index.php/eljc/article/view/v15i1r92.

Further reading: *Enumerative Combinatorics, Vol. 2,* by R.P. Stanley, CUP, 2001, Chapter 5 (exercises).

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- The number of parking functions on $\{1,\dots,n\}$ is $(n+1)^{n-1}$
- The number of rooted forests on n vertices is $(n+1)^{n-1}$
- The number of bases of $\mathbb{R}^n$ that can be made from $(0,\dots,0,1,\dots,1,0,\dots,0)$ is $(n+1)^{n-1}$

Thank you for your attention!

I hope that was of some help.