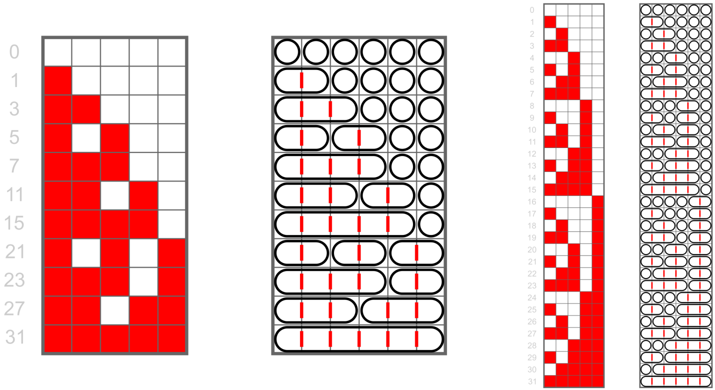
What are...multiplicative compositions?

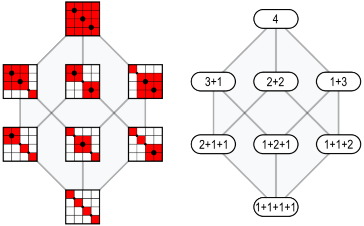
Or: Additive < multiplicative

Partitions and compositions



- (Integer) partition = an ordered way of writing an integer as a sum of positive integers, e.g. $4 = 2 + 1 + 1$
- (Integer) compositions = the same but without order, e.g. $4 = 1 + 2 + 1$
- Counting them is a classical topic in mathematics

Lets count them!



- **Proof without words** (above): the number of compositions of n is 2^{n-1}
- **More difficult**: the number of partitions $p(n)$ is

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \text{ as } n \to \infty$$ Like it!

In 1937, Hans Rademacher found a way to represent the partition function $p(n)$ by the convergent series

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\,\right]\right)$$

Do we like this one? Well..

where

$$A_k(n) = \sum_{0 \le m < k,\ (m,k)=1} e^{\pi i (s(m,k) - 2nm/k)}.$$

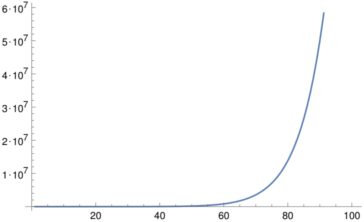
and $s(m,k)$ is the Dedekind sum.

We like the asymptotic $\sim$ formula!

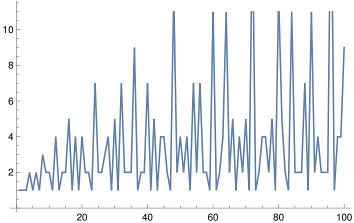
$f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$

Enter: Multiplication

#add. partitions:



#mult. partitions:

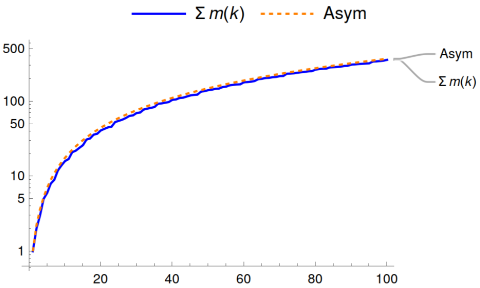


- **Multiplicative partition** = an ordered way of writing an integer as a product of positive integers ≥ 2, e.g. $12 = 3 \cdot 2 \cdot 2$
- **Multiplicative compositions** = the same but without order, e.g. $12 = 2 \cdot 3 \cdot 2$
- **Task** Count them!

Enter, the theorem

We get the following asymptotic formulas:

$$\sum_{k=1}^{n} m(k) \sim n \cdot \frac{1}{2\pi} \exp\left(2\sqrt{\ln(n)}\right) \ln(n)^{-3/4}$$



$$\sum_{k=1}^{n} M(k) \sim (0.311736521...)n^{1.7286472389...}$$

- $m(n)$ = mult. partitions, $M(n)$ = mult. compositions
- The exponent $\rho \approx 1.7286472389$ is the unique solution of $\zeta(x) = 2$ with $x > 1$

Where are they?

We know a lot about bacteria that grow nicely in a Petri plate; what about the rest?

- Additive partitions/compositions are everywhere in mathematics
- Multiplicative partitions/compositions are where precisely ?

Thank you for your attention!

I hope that was of some help.