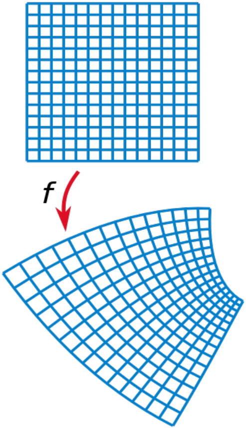
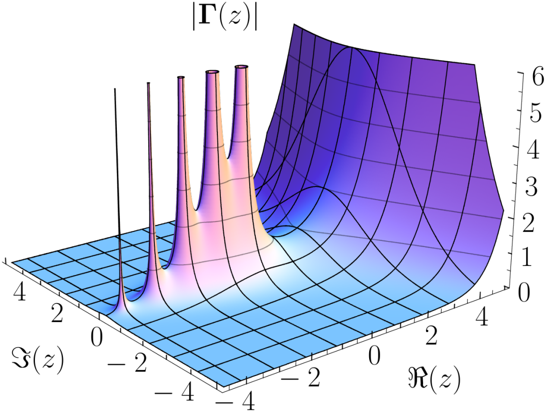
What is...the Riemann–Roch theorem?

Or: Allowing poles

Holomorphic versus meromorphic

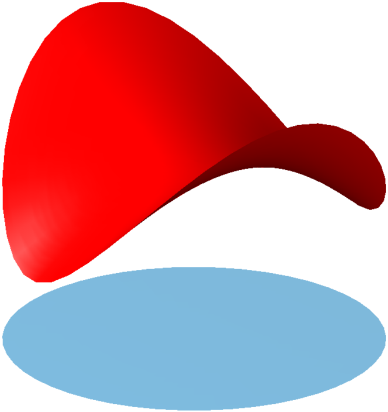




- Holomorphic function = complex analytic = polynomials, $\sin z$,...
- Meromorphic function = complex analytic up to isolated poles = holomorphic, rational functions, $1/\sin z$,...
- Question How different are these?

Maximum modulus principle (MMP)

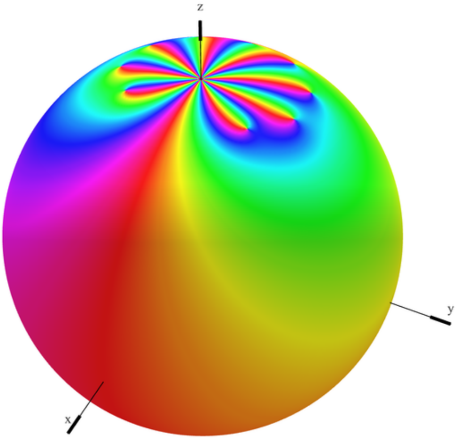
$|\cos(z)|$
on the
unit disc,
the max.
is not in
the interior:



- MMP A holomorphic $f\colon U \to \mathbb{C}$ cannot have a strict maximum on a connected open set $U \subset \mathbb{C}$
- This implies that the only holomorphic maps $S^2 \to \mathbb{C}$ are constants
- This is exciting and disappointing at the same time

Riemann's question

A pole at ∞:



- Previous slide There is essentially only one holomorphic map $S^2 \to \mathbb{C}$
- Question How many maps are there if we allow poles?
- Task Find a lower bound for #maps in terms of the number of poles

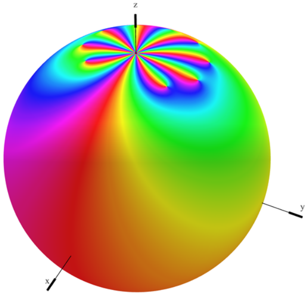
Enter, the theorem

Non-constant meromorphic functions exist, i.e.

$$\dim l(n_1, ..., n_r) \geq n_1 + ... + n_r + 1$$

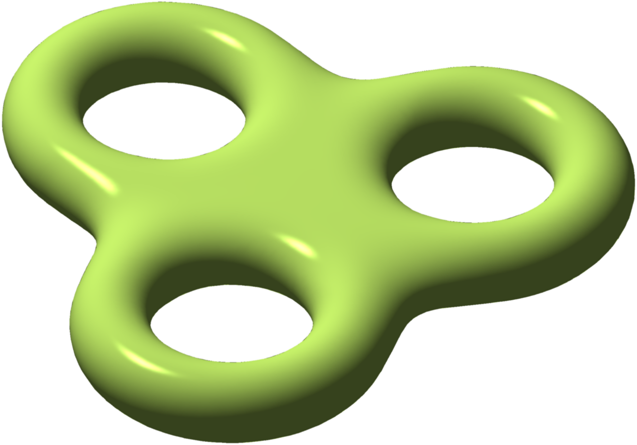
where $n_i \geq 0$

- Here $l(n_1, ..., n_r)$ = space of meromorphic functions $S^2 \to \mathbb{C}$ that are allowed to have r poles of order at most n_i
- Example If $r = 1$ and $n_1 = 1$ then we get $\dim l(1) \geq 2$, so there is a non-constant function



Three generalizations

genus $= 3$:



- Allow negative n_i Then we can also count zeros of functions
- Allow other surfaces Then $\dim l(n_1, \ldots, n_r) \geq n_1 + \ldots + n_r - g + 1$
- Roch There is also an equality using a correction term

Thank you for your attention!

I hope that was of some help.