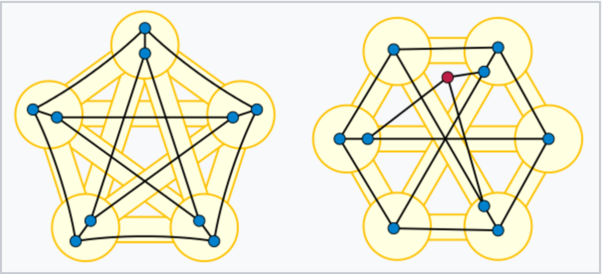
What are...intrinsically linked graphs?

Or: Difficult problem, easy solution

(Kuratowski–)Wagner theorem

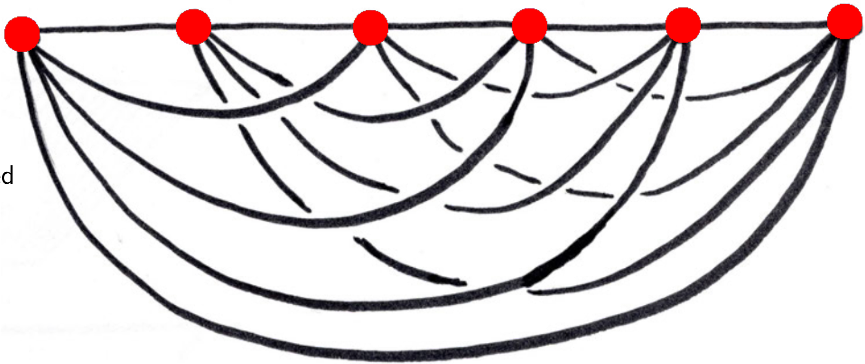


K_5 (left) and $K_{3,3}$ (right) as minors of the nonplanar Petersen graph (small colored circles and solid black edges). The minors may be formed by deleting the red vertex and contracting edges within each yellow circle.

- Planar graph = can be drawn in the plane without intersection
- Theorem $K_{3,3}$ and K_5 are witnesses for planarity testing
- Consequence Recognizing planarity is in $O(n^3)$ for $n = \#\text{vertices}$

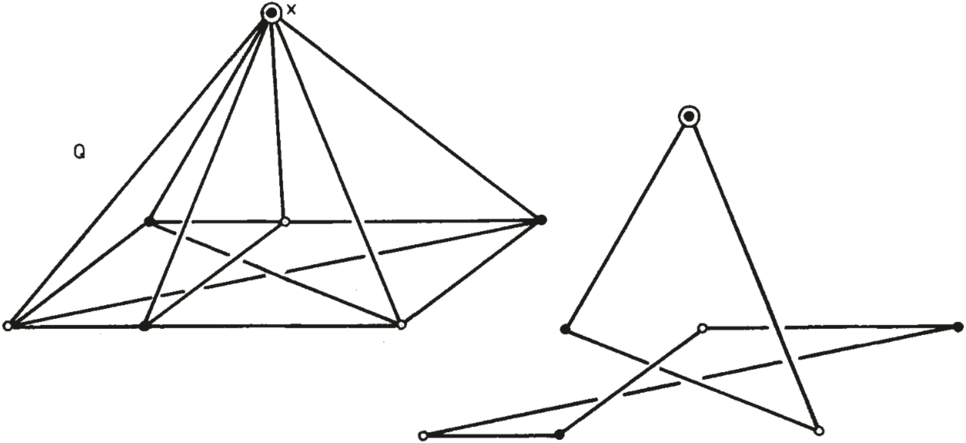
Knotted graphs

A knotted K_6:



- **Fact** Every graph can be embedded in $\mathbb{R}^3$
- **Question** Can one draw a graph in 3-space $\mathbb{R}^3$ without knotting?
- **Unclear!?** Can anything be said?

$K_{3,3,1}$ is intrinsically linked (IL)

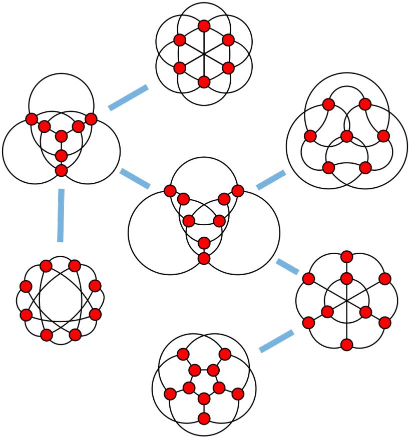


- IL = every embedding contains a non-trivial link
- Proof of headline $K_{3,3,1}$ is IL $\Leftrightarrow$ $K_{3,3}$ is not planar
- Recall $K_{3,3}$ is indeed not planar

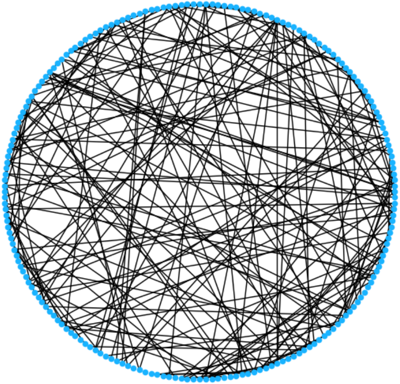
Enter, the theorem

IL testing is in $O(n^3)$

- This bound is quite naive – one can likely do better
- Theorem There are only seven witnesses:



"All" graphs are IL



- Fact Random graphs have many edges $\Rightarrow$ likely IL
- "Most" graphs are random
- Theorem Almost all graphs are IL

Thank you for your attention!

I hope that was of some help.