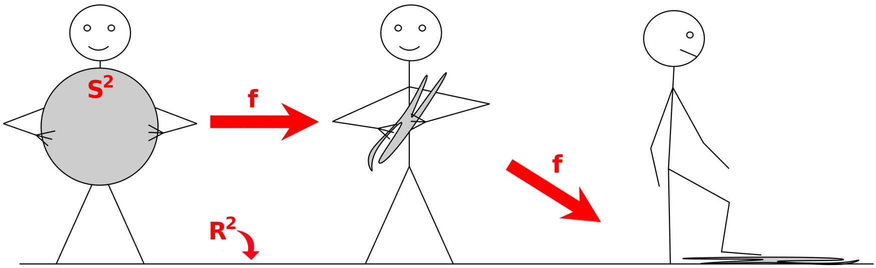
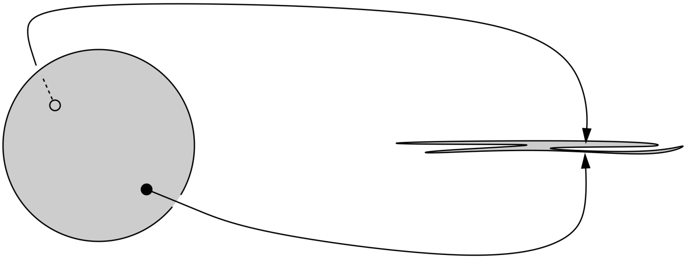
What are...incarnations of the Borsuk–Ulam theorem?

Or: Topology or combinatorics or...?

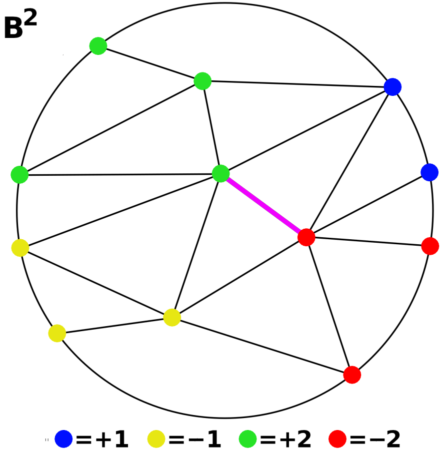
Borsuk–Ulam topologically

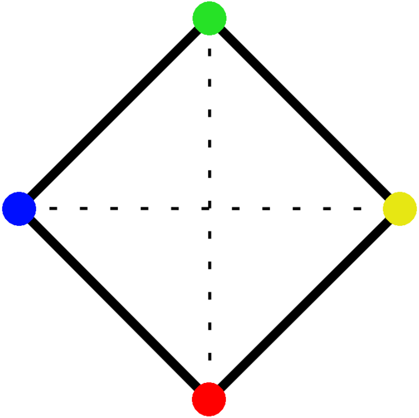




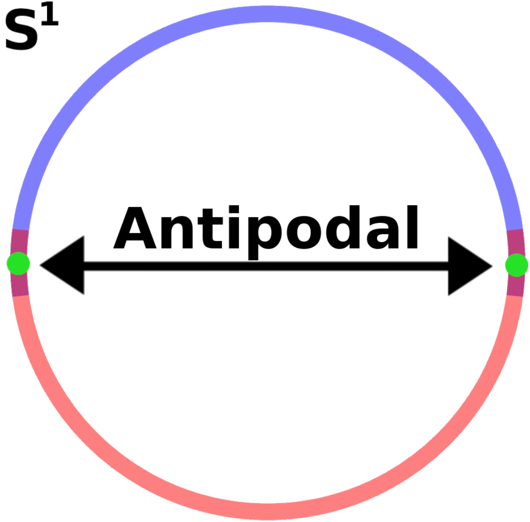
For every continues map $f\colon S^n \to \mathbb{R}^n \ \exists x\colon f(x) = f(-x)$ Topology

Borsuk–Ulam combinatorially





Antipodally triangulate B^n, label vertices $\{\pm 1, \ldots, \pm n\}$ in antipodal-symmetric fashion, then $\exists$ edge with vertices labeled $(x, -x)$ Combinatorics



Open covering $S^n = \bigcup_{i=0}^{n} U_i$, then at least one U_i contains some $x, -x$ Covering

Enter, the theorem

The following are equivalent and true

(a) For every continues map $f\colon S^n \to \mathbb{R}^n$ $\exists x\colon f(x) = f(-x)$ Topology

(a') For every antipodal map $f\colon S^n \to \mathbb{R}^n$ (*i.e.* continues and $f(-x) = -f(x)$) $\exists x\colon f(x) = 0$

(A) There is no antipodal map $f\colon S^n \to S^{n-1}$

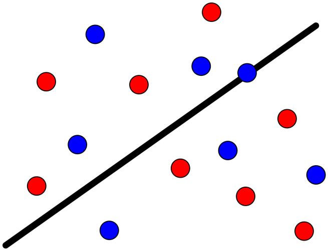
(A') There is no continues map $f\colon B^n \to S^{n-1}$ that is antipodal on the boundary

(b) Antipodally triangulate B^n, label vertices $\{\pm 1, \dots, \pm n\}$ in antipodal-symmetric fashion, then $\exists$ edge with vertices labeled $(x, -x)$ Combinatorics

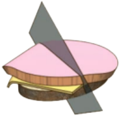
(c) Open covering $S^n = \bigcup_{i=0}^{n} U_i$, then at least one U_i contains some $(x, -x)$ Covering

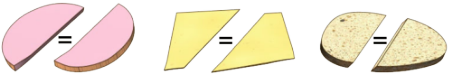
(a), (a'), (A), (A') are due to Borsuk (conjectured by Ulam), (b) is Tucker's lemma, (c) is the Lyusternik–Shnirel'man theorem

Ham sandwiches



For a finite set of blue or red colored points in $\mathbb{R}^2$ there is a line that simultaneously bisects the red points and bisects the blue points Combinatorics





n measurable sets in $\mathbb{R}^n$ can be divided in half by one single hyperplane Covering

Thank you for your attention!

I hope that was of some help.