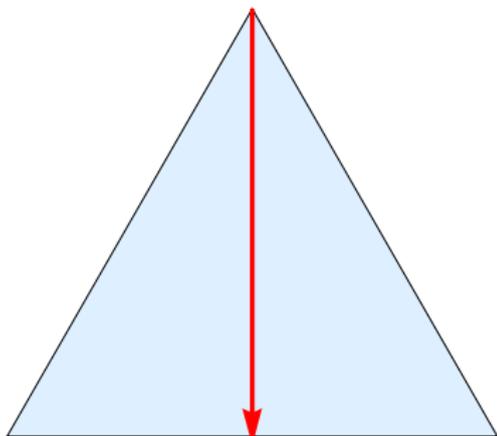


What is...the finite Kakeya problem?

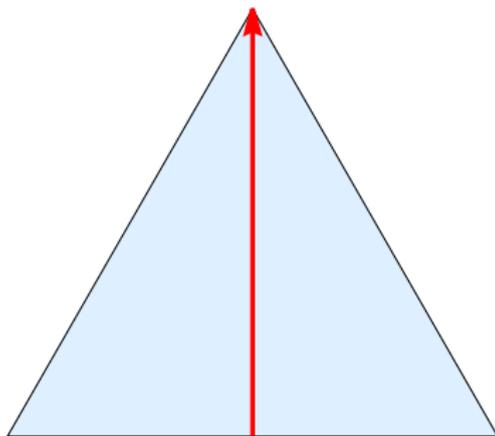
Or: A finite filling

The classical Kakeya problem

Start:



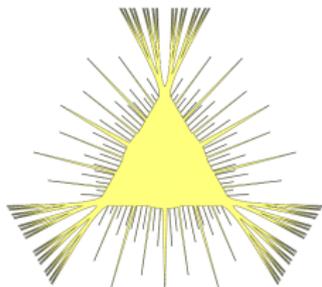
Finish:



- ▶ **Kakeya's problem** What is a minimum area of a region D in the plane, in which a needle of unit length can be turned through 180 degree?
- ▶ If D is assumed to be convex, then D is an equilateral triangle **Relatively easy**
- ▶ In general, the area of D can be arbitrary small **Strange**

The Kakeya conjecture

- ▶ A Kakeya set $K \subset \mathbb{R}^n$ is a set such that a unit line segment can be rotated continuously through 180 degrees within it



Kakeya sets can have arbitrary

small volume > 0

- ▶ A Besicovitch set $B \subset \mathbb{R}^n$ contains a unit line segment in every direction



$n = 1$



$n = 2$



$n = 4$



$n = 256$

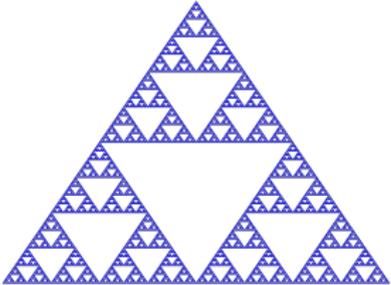
Besicovitch sets can

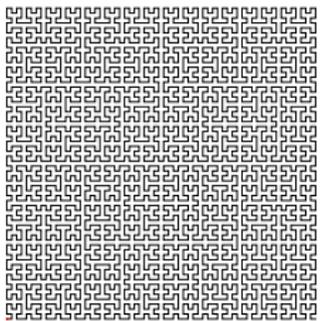
have volume $= 0$

Kakeya conjecture Besicovitch sets have Hausdorff/Minkowski dimension n

Filling space

The Hausdorff dimension hd is a measure of how space filling an object is, e.g.

$$hd \left(\begin{array}{c} \text{Sierpinski Triangle} \end{array} \right) = \ln(3)/\ln(2) \approx 1.6$$
A Sierpinski triangle fractal, a self-similar geometric figure. It is constructed by starting with a large equilateral triangle, dividing it into four smaller equilateral triangles, and removing the central one. This process is repeated infinitely, resulting in a fractal with a Hausdorff dimension of approximately 1.6.

$$hd \left(\begin{array}{c} \text{Space-filling curve} \end{array} \right) = 2$$
A space-filling curve, a fractal curve that fills a square. It is a continuous curve that passes through every point in the square, despite having a Hausdorff dimension of 2. The curve is highly convoluted and self-similar.

Conjecture reformulated B may have volume zero, but still fills space

Enter, the theorem

A finite Besicovitch set B is a subset of \mathbb{F}_q^n for a finite field \mathbb{F}_q of order $|\mathbb{F}_q| = q$ that contains a line in every direction, *i.e.*

$$\forall x \in \mathbb{F}_q^n \exists y \in \mathbb{F}_q^n : L = \{y + a \cdot x \mid a \in \mathbb{F}_q\} \subset B$$

Finite Kakeya conjecture Is there a constant c , only depending on n , such that every B satisfies

$$|B| \geq cq^n?$$

- ▶ Theorem (Dvir) ~2008. The conjecture is true
- ▶ The proof uses only combinatorics of polynomials and is short
- ▶ The original Kakeya conjecture is (wildly) open (in 2021)

A glimpse at the proof

- ▶ Lemma 1 (Schwartz–Zippel). Every non-zero polynomial $f \in \mathbb{F}_q[X_1, \dots, X_n]$ of degree d has at most dq^{n-1} roots in \mathbb{F}_q^n
 - ▶ Lemma 2. For every set $E \subset \mathbb{F}_q^n$ of size $|E| < \binom{n+d}{d}$ there is a non-zero polynomial $f \in \mathbb{F}_q[X_1, \dots, X_n]$ of degree at most d that vanishes on E
-

These are **generalizations** of the well-known facts:

- ▶ Lemma 1'. Every polynomial of degree d in one variable has at most d roots

$$\text{worst-case: } f = (X - a_1) \dots (X - a_d)$$

- ▶ Lemma 2'. For every set $E = \{a_1, \dots, a_r\} \subset \mathbb{F}_q$ of size $|E| \leq d$ there is a non-zero polynomial of degree at most d that vanishes on E

$$\text{take: } f = (X - a_1) \dots (X - a_r)$$

Thank you for your attention!

I hope that was of some help.