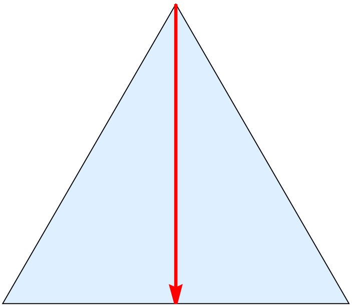
What is...the finite Kakeya problem?

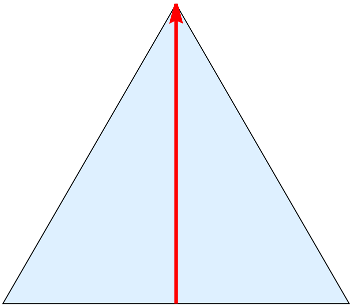
Or: A finite filling

The classical Kakeya problem

Start:



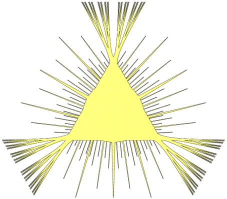
Finish:



- Kakeya's problem What is a minimum area of a region D in the plane, in which a needle of unit length can be turned through 180 degree?
- If D is assumed to be convex, then D is an equilateral triangle Relatively easy
- In general, the area of D can be arbitrary small Strange

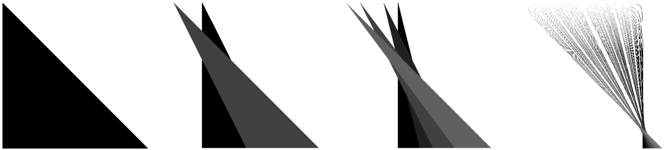
The Kakeya conjecture

- A Kakeya set $K \subset \mathbb{R}^n$ is a set such that a unit line segment can be rotated continuously through 180 degrees within it



Kakeya sets can have arbitrary small volume > 0

- A Besicovitch set $B \subset \mathbb{R}^n$ contains a unit line segment in every direction



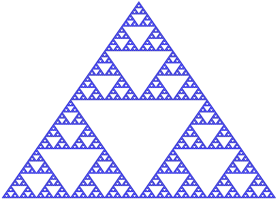
$n = 1$ $n = 2$ $n = 4$ $n = 256$

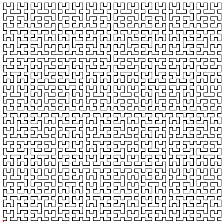
Besicovitch sets can have volume $= 0$

Kakeya conjecture Besicovitch sets have Hausdorff/Minkowski dimension n

Filling space

The Hausdorff dimension hd is a measure of how space filling an object is, *e.g.*

hd () $= \ln(3)/\ln(2) \approx 1.6$

hd () $= 2$

Conjecture reformulated B may have volume zero, but still fills space

Enter, the theorem

A finite Besicovitch set B is a subset of $\mathbb{F}_q^n$ for a finite field $\mathbb{F}_q$ of order $|\mathbb{F}_q| = q$ that contains a line in every direction, *i.e.*

$$\forall x \in \mathbb{F}_q^n \exists y \in \mathbb{F}_q^n : L = \{y + a \cdot x \mid a \in \mathbb{F}_q\} \subset B$$

Finite Kakeya conjecture Is there a constant c, only depending on n, such that every B satisfies

$$|B| \geq cq^n?$$

- Theorem (Dvir) $\sim$2008. The conjecture is true
- The proof uses only combinatorics of polynomials and is short
- The original Kakeya conjecture is (wildly) open (in 2021)

A glimpse at the proof

- Lemma 1 (Schwartz–Zippel). Every non-zero polynomial $f \in \mathbb{F}_q[X_1, ..., X_n]$ of degree d has at most dq^{n-1} roots in $\mathbb{F}_q^n$
- Lemma 2. For every set $E \subset \mathbb{F}_q^n$ of size $|E| < \binom{n+d}{d}$ there is a non-zero polynomial $f \in \mathbb{F}_q[X_1, ..., X_n]$ of degree at most d that vanishes on E

These are generalizations of the well-known facts:

- Lemma 1'. Every polynomial of degree d in one variable has at most d roots

worst-case: $f = (X - a_1)...(X - a_d)$

- Lemma 2'. For every set $E = \{a_1, ..., a_r\} \subset \mathbb{F}_q$ of size $|E| \leq d$ there is a non-zero polynomial of degree at most d that vanishes on E

take: $f = (X - a_1)...(X - a_r)$

Thank you for your attention!

I hope that was of some help.