What is...the Mason–Stothers theorem?

Or: The ABC of polynomials

ABC in number theory

$a + b = c$ coprime, $\mathrm{rad}(a, b, c)$ =product of distinct prime factors, then

$$(*)\ c \leq \mathrm{rad}(a, b, c) \text{ almost always}$$

There are $\sim$24 million triples not satisfying $(*)$ among all triples with $c < 10^{18}$, *e.g.*

$$a = 7168, b = 78125, c = 85293$$

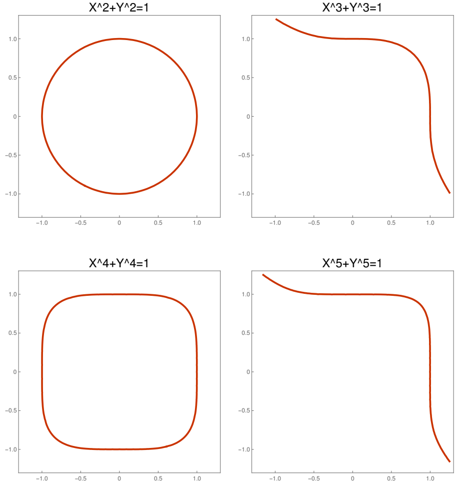
$a + b = c$: $2^{10} \cdot 7 + 5^7 = 3^8 \cdot 13$

$\mathrm{rad}(a, b, c) = \mathrm{rad}(2^{10} \cdot 3^8 \cdot 5^7 \cdot 7 \cdot 13) = 2730 < c$

$\mathrm{rad}(a, b, c) / c = \frac{70}{2187} = 0.0320073 < 1$

$q(a, b, c) = q(7168, 78125, 85293) = 1.43501 > 1$

Fermat in number theory



There are no non-trivial rational points on $X^n + Y^n = 1$ (Fermat curve) for $n > 2$

Fermat (known but famously difficult to prove) "follows directly from" ABC

Lets move to polynomials

$$a = (X+1)\cdot(X+2), b = 2X^4, c = a+b$$
$$c = 2X^4 + X^2 + 3X + 2$$
$$\mathrm{rad}(a,b,c) = (X+1)\cdot(X+2)\cdot X\cdot(2X^4 + X^2 + 3X + 2)$$

Note that the degree of c is lower than the degree of $\mathrm{rad}(a,b,c)$:

$$4 = \deg(c) \leq \deg(\mathrm{rad}(a,b,c)) = 7$$

$$a = (X+1)^3\cdot(X+2)^5, b = 2X^4, c = a+b$$
$$c = X^8 + 13X^7 + 73X^6 + 231X^5 + 452X^4 + 552X^3 + 416X^2 + 176X + 32$$
$$\mathrm{rad}(a,b,c) = (X+1)\cdot(X+2)\cdot X\cdot(X^8 + \mathrm{REST})$$

Note that the degree of c is lower than the degree of $\mathrm{rad}(a,b,c)$:

$$8 = \deg(c) \leq \deg(\mathrm{rad}(a,b,c)) = 11$$

Enter, the theorem

$a+b=c$ coprime polynomial in $\mathbb{R}_{\geq 0}[X]$, not all constant, $\mathrm{rad}(a,b,c)$ =product of distinct irreducible factors, then

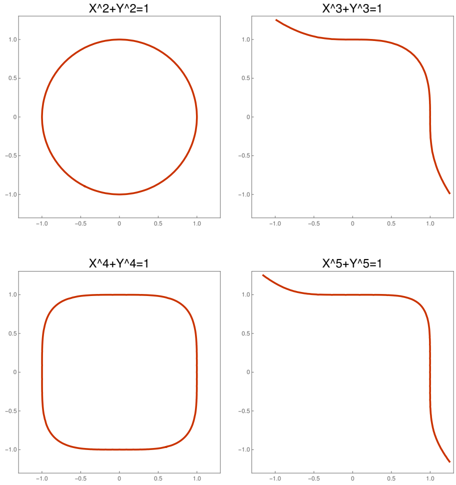
$$\deg(c) \leq \deg(\mathrm{rad}(a,b,c)) \text{ always}$$

- Actually we can write $\deg(\mathrm{rad}(a,b,c)) - 1$ on the right-hand side; this is a sharp bound:

$$a = X^4, b = (2X+1)\cdot(2X^2+2X+1), c = (X+1)^4$$
$$4 = \deg(c) \leq \deg(\mathrm{rad}(a,b,c)) = \deg(X\cdot(2X+1)\cdot(2X^2+2X+1)\cdot(X+1)) = 5$$

- There are various versions of this theorem; an appropriate formulation holds over $\mathbb{K}[X]$ for any field $\mathbb{K}$
- There is a slick proof of this theorem which is roughly $1/2$ of a page long

Fermat in polynomials



There are no non-trivial $\mathbb{Q}(t)$-rational points on $\boxed{X^n + Y^n = 1}$ for $n > 2$

Polynomial Fermat follows directly from polynomial ABC

Thank you for your attention!

I hope that was of some help