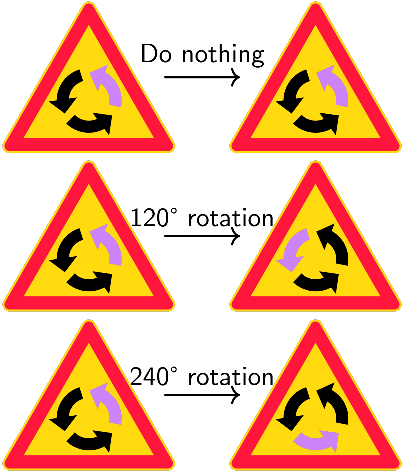
What is...a Coxeter group?

Or: What is...$1, \infty, 3, 5, 3, 4, 4, 4, 3, 3, 3, 3, ...$?

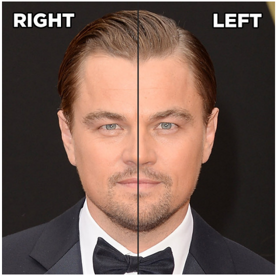
Enter, symmetry



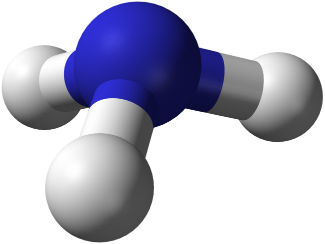
- A symmetry is an operation that does not change the object
- Mathematically, these form a certain algebraic structure called a group

Symmetry is everywhere

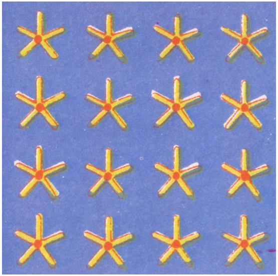
Human face – $\mathbb{Z}/2\mathbb{Z}$ symmetry



Ammonia – S_3 symmetry

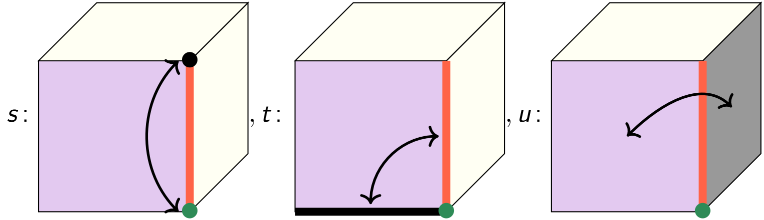


Tomb in egypt – Translations+reflections



Earth – ∞ many symmetries

Symmetries of a regular polygon P



$$\angle(s,t) = \cos(2\pi/\boxed{4}), \quad \angle(t,u) = \cos(2\pi/\boxed{3}), \quad \angle(s,u) = \cos(2\pi/\boxed{2})$$

$$m(s,t) = \boxed{4}, \quad m(t,u) = \boxed{3}, \quad m(s,u) = \boxed{2}$$

$$\Gamma = \; s \overset{4}{\text{———}} t \text{———} u$$

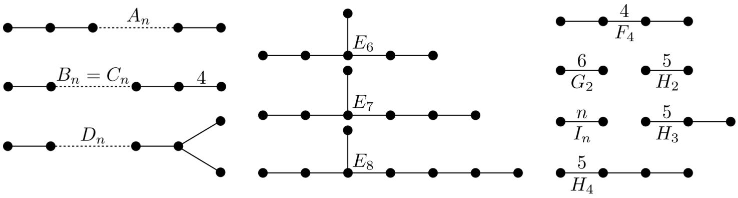
- For a flag in P there are associated reflections s, t, u
- The group of symmetries G of P admits the presentation

$$G \cong \langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^{m(s,t)} = (tu)^{m(t,u)} = (su)^{m(s,u)} = 1 \rangle$$

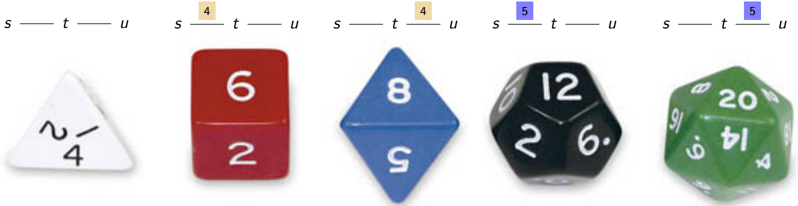
- This datum is determined by a graph Γ (edges 2 and labels 3 are omitted)

Enter, the theorem

A group generated by reflections is finite if and only if Γs components are of the form



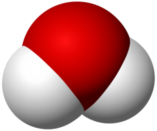
- This classifies the finite reflection symmetries
- This generalizes Platonic solids: non-branching graphs ⇝ regular polygons



Water, ammonia and methane

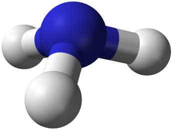
Water – S_2 symmetry

$A_1 \leftrightsquigarrow \quad s \quad \leftrightsquigarrow$



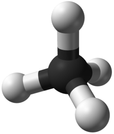
Ammonia – S_3 symmetry

$A_2 \leftrightsquigarrow \quad s \text{ ——— } t \quad \leftrightsquigarrow$



Methane – S_4 symmetry

$A_3 \leftrightsquigarrow \quad s \text{ ——— } t \text{ ——— } u \quad \leftrightsquigarrow$



- Coxeter groups of type A are symmetric groups
- So Coxeter groups also generalize symmetric groups

Thank you for your attention!

I hope that was of some help.