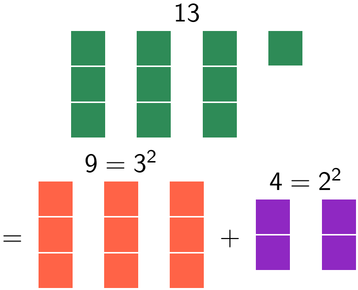
What is...the fifteen theorem?

Or: What is special about 1,2,3,5,6,7,10,14,15?

Fermat's two squares

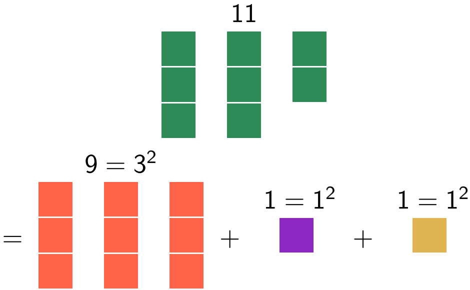


- (An odd prime p can be written as $p = a^2 + b^2$) $\Leftrightarrow$ (4 divides $p - 1$)
- We miss numbers!

A001481 Numbers that are the sum of 2 squares.
(Formerly M0968 N0361)

0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50, 52, 53, 58, 61, 64, 65, 68, 72, 73, 74, 80, 81, 82, 85, 89, 90, 97, 98, 100, 101, 104, 106, 109, 113, 116, 117, 121, 122, 125, 128, 130, 136, 137, 144, 145, 146, 148, 149, 153, 157, 160 (list; graph; refs; listen; history; text; internal format)

Legendre's three-squares

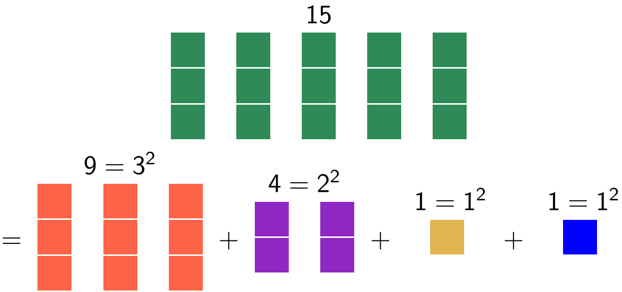


- $\left(n \text{ can be written as } n = a^2 + b^2 + c^2\right) \Leftrightarrow n \neq 4^x(8y + 7)$
- We miss numbers!

A004215 Numbers that are the sum of 4 but no fewer nonzero squares.
(Formerly M4349)

7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79, 87, 92, 95, 103, 111, 112, 119, 124, 127, 135, 143, 151, 156, 159, 167, 175, 183, 188, 191, 199, 207, 215, 220, 223, 231, 239, 240, 247, 252, 255, 263, 271, 279, 284, 287, 295, 303, 311, 316, 319, 327, 335, 343 (list; graph; refs; listen; history; text;

Lagrange's four-squares



- n can always be written as $n = a^2 + b^2 + c^2 + d^2$
- We get all numbers!

A000118 Number of ways of writing n as a sum of 4 squares; also theta series of lattice Z^4.

1, 8, 24, 32, 24, 48, 96, 64, 24, 104, 144, 96, 96, 112, 192, 192, 24, 144, 312, 160, 144, 256, 288, 192, 96, 248, 336, 320, 192, 240, 576, 256, 24, 384, 432, 384, 312, 304, 480, 448, 144, 336, 768, 352, 288, 624, 576, 384, 96, 456, 744, 576, 336, 432, 960, 576, 192 (list;

Enter, the theorem

Let A be a positive-definite quadratic form defined by an integral matrix

A is universal (it takes all values in $\mathbb{N}$)

$\Leftrightarrow$

A takes the values $1, 2, 3, 5, 6, 7, 10, 14, 15$

	$w \downarrow$	$x \downarrow$	$y \downarrow$	$z \downarrow$
$w \rightarrow$	1			
$x \rightarrow$		1		
$y \rightarrow$			1	
$z \rightarrow$				1

$w^2 + x^2 + y^2 + z^2$

w	x	y	z	
0	0	0	1	**1**
0	0	1	1	**2**
0	1	1	1	**3**
0	0	1	2	**5**
0	1	1	2	**6**
1	1	1	2	7
1	1	2	2	**10**
0	1	2	3	**14**
1	1	2	3	**15**

Lagrange's four squares

	$w \downarrow$	$x \downarrow$	$y \downarrow$	$z \downarrow$
$w \rightarrow$	1	0	1	2
$x \rightarrow$	0	2	0	-2
$y \rightarrow$	1	0	2	0
$z \rightarrow$	2	-2	0	11

$w^2 + 2x^2 + 2y^2 + 11z^2$

$+4wz$ $+2wy$ $-4xz$

w	x	y	z	
1	0	0	0	**1**
0	0	1	0	**2**
1	1	0	0	**3**
1	0	1	0	**5**
1	-1	0	-1	**6**
1	1	1	0	7
0	1	2	0	**10**
1	1	0	1	**14**
1	1	2	0	**15**

Another universal form

What if the off-diagonals are not divisible by 2?

Let A be a positive-definite integral quadratic form

A is universal (it takes all values in $\mathbb{N}$)

$\Leftrightarrow$

A takes the values $1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22$
$23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290$

- The above also takes forms such as $x^2 + xy + y^2$ into account

$$x^2 + xy + y^2 \leftrightsquigarrow \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

- There are 54 universal four-variable diagonal quadratic forms
- There are 6436 universal four-variable quadratic forms

Thank you for your attention!

I hope that was of some help.