What is...the philosophy of generating functions?

Or: How to encode counting problems

Natural numbers as coefficients

$$g_1(z) = \frac{1}{1-z} = 1\,z^0 + 1\,z^1 + 1\,z^2 + 1\,z^3 + 1\,z^4 + 1\,z^5 + \ldots$$

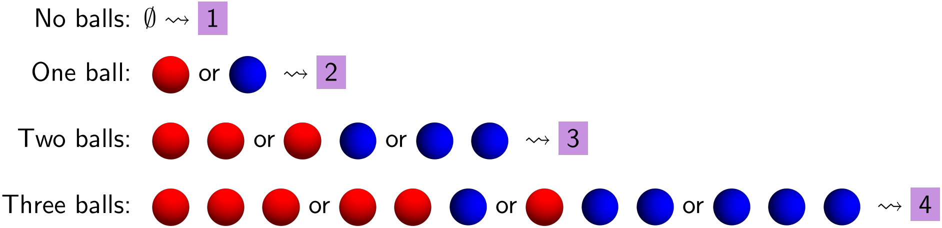
$$g_2(z) = \frac{1}{1-z^2} = 1\,z^0 + 0\,z^1 + 1\,z^2 + 0\,z^3 + 1\,z^4 + 0\,z^5 + \ldots$$

$$g_3(z) = \frac{1-z^2}{1-z} = 1\,z^0 + 1\,z^1 + 0\,z^2 + 0\,z^3 + 0\,z^4 + 0\,z^5 + \ldots$$

$$g_4(z) = \frac{1}{1-z}\frac{1}{1-z} = 1\,z^0 + 2\,z^1 + 3\,z^2 + 4\,z^3 + 5\,z^4 + 6\,z^5 + \ldots$$

We are counting something!

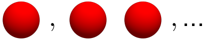
Ways to select balls?



The number of ways to select k balls is the coefficient of z^k in $g_4(z) = \frac{1}{1-z}\frac{1}{1-z}$. So $g_4(z)$ generates the number of ways to select balls out of two colors

More ball counting

$$g_1(z) = \frac{1}{1-z} \rightsquigarrow \emptyset,$$



$$g_2(z) = \frac{1}{1-z^2} \rightsquigarrow \emptyset,$$

$$g_3(z) = \frac{1-z^2}{1-z} \rightsquigarrow \emptyset,$$

$$g_4(z) = \frac{1}{1-z}\frac{1}{1-z} \rightsquigarrow \emptyset,$$

- $g_1(z)$ encodes the number of ways to select k balls from one color
- $g_2(z)$ encodes the number of ways to select $2k$ balls from one color
- $g_3(z)$ encodes the number of ways to select at most one ball from one color
- $g_4(z)$ encodes the number of ways to select k balls from two colors

The functions $g_i(z)$ are an efficient way to encode these counting problems

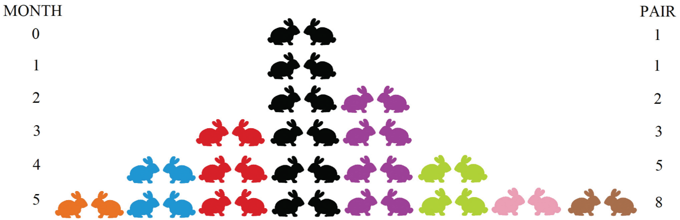
Enter, the theorem/philosophy!

A generating function is a way of encoding an infinite sequence of numbers by treating them as the coefficients of a formal power series.

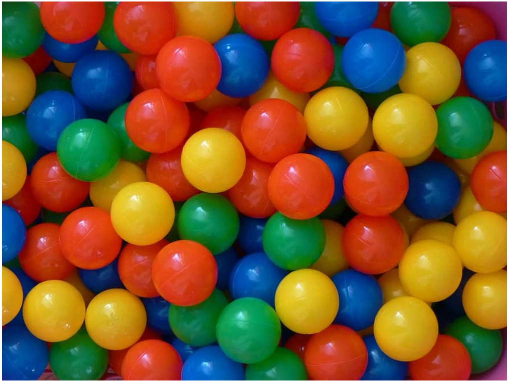
A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag (Pólya)

The rabbit counting a.k.a. Fibonacci numbers:

$$g(z) = \frac{1}{1 - z - z^2} = 1\,z^0 + 1\,z^1 + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + \dots$$



More balls



The number of ways to select 100 balls colored red, blue, yellow and green is the coefficient of z^{100} in

$$g(z) = \frac{1}{1-z} \cdot \frac{1}{1-z} \cdot \frac{1}{1-z} \cdot \frac{1}{1-z}$$

Thank you for your attention!

I hope that was of some help.