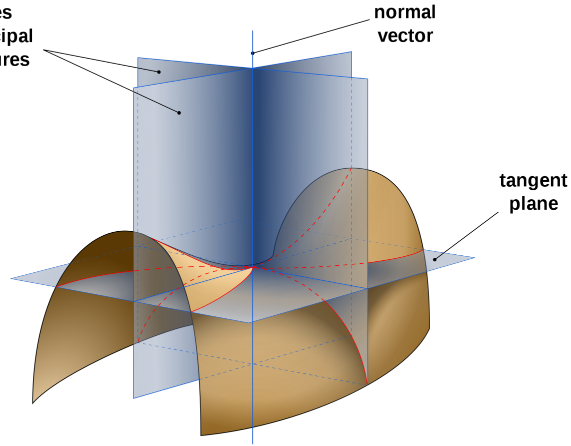
What is...the Gauss–Bonnet theorem?

Or: Curvature and Euler

Curvature in a nutshell

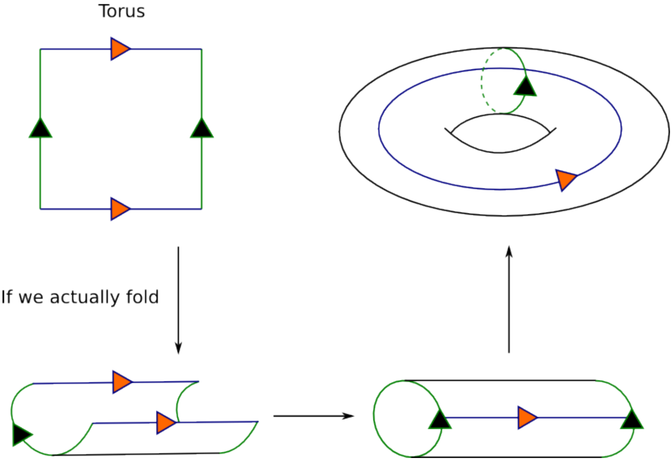
negative curvature:



- Normal planes intersect a surfaces in curves
- Take the minimum k_1 and maximum k_2 of the curvature of these curves
- $K = k_1 \cdot k_2$ Gaussian curvature

Euler characteristic in a nutshell

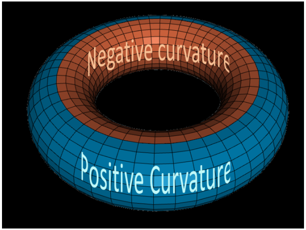
$\chi = 0$:



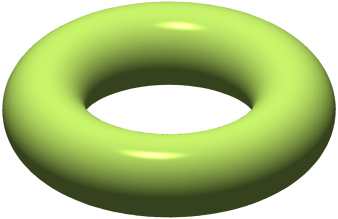
- Take a cell structure of a surface a.k.a. “a generalized triangulation”
- Say it has V vertices, E edges and F faces
- $\chi = V - E + F$ Euler characteristic

These have obviously nothing in common!

zero curvature:



Euler char zero:



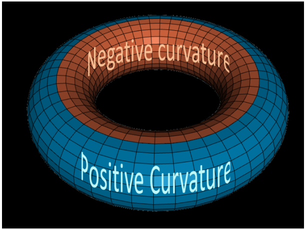
- Curvature is part of differential geometry
- Euler characteristic is part of combinatorics
- The Gauss–Bonnet theorem relates them

Enter, the theorem

Suppose M is a compact 2d Riemannian manifold without boundary, then

$$\int_M K dA = 2\chi(M)$$

- dA is the element of area of M
- Upshot χ is a topological invariant, thus $\int_M K dA$ is!
- There is also a version with boundary
- There are discrete versions as well
- This implies for example that the total curvature of a torus is zero



Gauss–Bonnet for graphs?

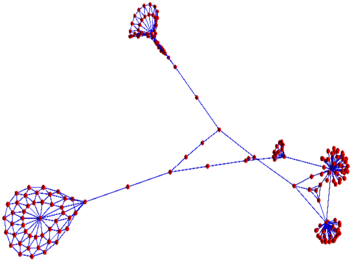


FIGURE 1. The sum of the curvature $K(v)$ is the Euler characteristic. The graph shown here has dimension 32549/20580 and Euler characteristic -4. The curvatures $-21, -31/2, -19/6, -5/3, -3/2, -1, 1/4$ appear once, $-1/2$ six times, $-1/4$ 3 times, 60 vertices have zero curvature, 50 have curvature 1/6 and 70 have curvature 1/2. These locally computed quantities add up to -4.

- There is a notion of curvature for vertices of graphs
- Gauss–Bonnet $\sum_{vertices} K(v) = \chi(G)$

Thank you for your attention!

I hope that was of some help.