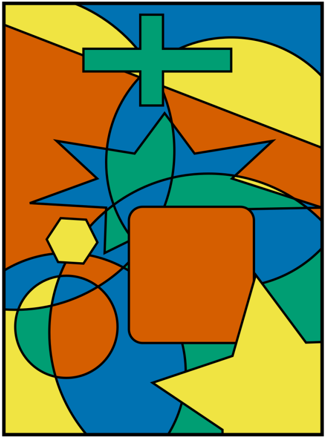
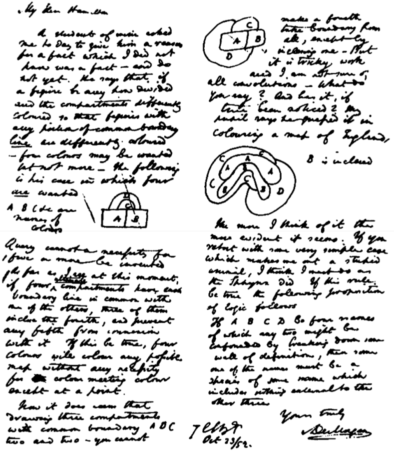
What is...Hadwiger's conjecture?

Or: Coloring is difficult...

Four colors suffice





My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured – four colours may be wanted but not more – the following is his case in which four are wanted

A B C &c are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two – you cannot make a fourth take boundary from all, except by inclosing one – But it is tricky work and I am not sure of all convolutions – What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

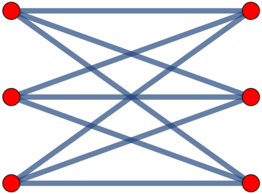
7 Chest Ter
Oct 23/52.

Yours truly
A De Morgan

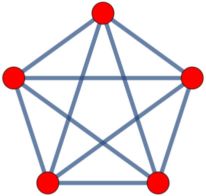
- Four color theorem (4CT) Any map can be colored with four colors
- Proposes in 1852 when Guthrie tried to color the map of counties of England, proved in 1976 by Appel–Haken
- The proof is one of the main achievements in graph theory

A different perspective on the 4CT

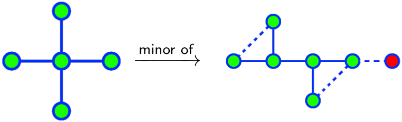
Complete bipartite graph on 6 vertices $K_{3,3}$



Complete graph on 5 vertices K_5

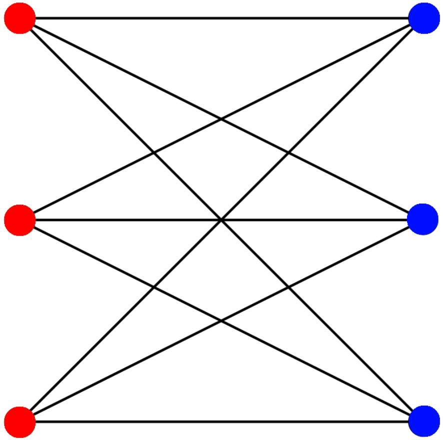


Wagner – top to bottom A graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor



- Wagner's theorem A graph is planar $\Leftrightarrow$ no $K_{3,3}$ or K_5 minors
- Coloring = adjacent vertices get different colors
- 4CT reformulated(?) A graph with at most a K_4 minor is 4-colorable

Where is $K_{3,3}$?



- Not that $K_{3,3}$ is bipartite = 2-colorable
- Any graph with $\geq$ one edge needs at least 2 colors
- Hence, $K_{3,3}$ should not play any role for the colorability

Enter, the theorem(s)

Conjecture (1943)

If G is loopless and has no K_t minor then its chromatic number is $< t$

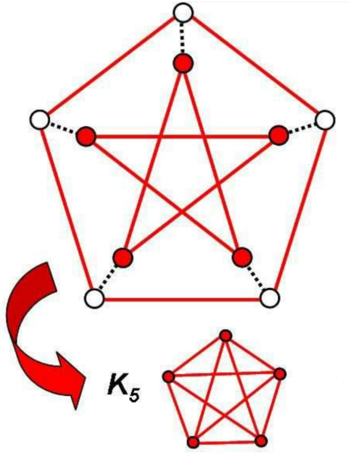
- **Theorem** The case $t = 5$ is true (4CT proven 1976)
- **Theorem** The case $t = 6$ is true (proven in 1993)
- $t > 6$ is open, but: **Theorem** The conjecture is almost always true:

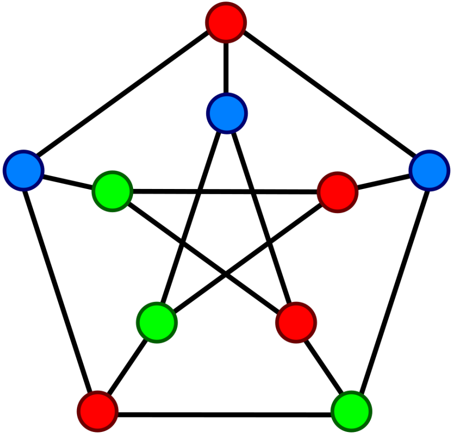
Hadwiger's Conjecture is True for Almost Every Graph

B. Bollobás, P. A. Catlin* and P. Erdös

The contraction clique number ccl(G) of a graph G is the maximal r for which G has a subcontraction to the complete graph K^r. We prove that for $d > 2$, almost every graph of order n satisfies $n((\log_2 n)^{\frac{1}{2}}+4)^{-1} \leqslant \mathrm{ccl}(G) \leqslant n(\log_2 n - d \log_2 \log_2 n)^{-\frac{1}{2}}$. This inequality implies the statement in the title.

The bound need not to be sharp





- The Petersen graph contains a K_5 minor
- The Petersen graph is 3-colorable

Thank you for your attention!

I hope that was of some help.