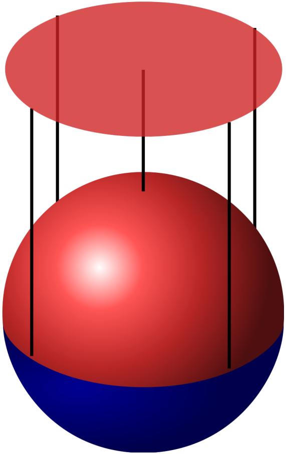
What is...Whitney's embedding theorem?

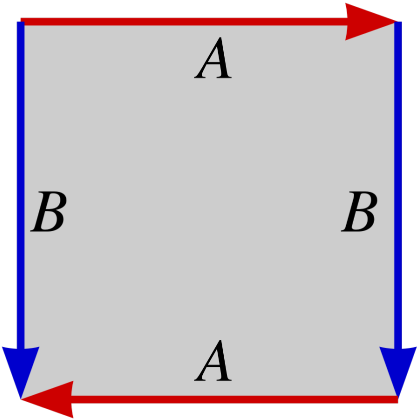
Or: Projective spaces are nasty

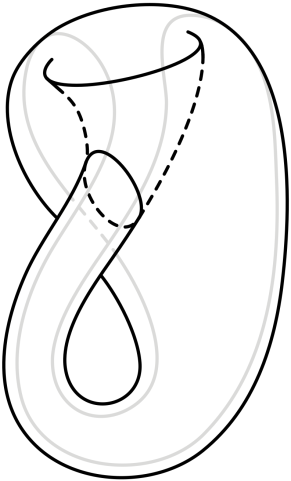
Manifolds = many discs



- Manifold (mfd) = discs glued together
- Examples are sphere, torus, pair of pants, ...
- Question Given an n-manifold, what is the smallest $\mathbb{R}^k$ it lives in?

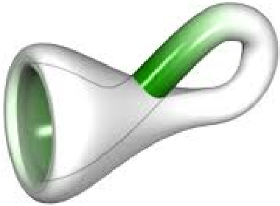
The Klein bottle

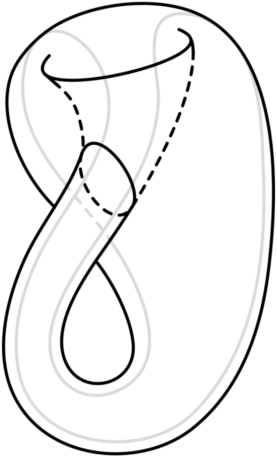




- Sphere, torus, pair of pants clearly live in $\mathbb{R}^3$ Maybe you have some at home!
- The Klein bottle is a surface (2d mfd) obtained from a square by identifying edges
- The Klein bottle can not be embedded in $\mathbb{R}^3$, but can be immersed

Colors are distinct





- Think of the fourth dimension as color
- Then the Klein bottle does not intersect itself
- In other words, the Klein bottle lives in 4d

Enter, the theorem

Let M be a smooth n-manifold for $n \geq 2$, then:

(i) M can be smoothly embedded in $\mathbb{R}^{2n}$ Lives in $2n$ dim

(ii) M can be immersed in $\mathbb{R}^{2n-1}$ It shadow lives in $2n-1$ dim

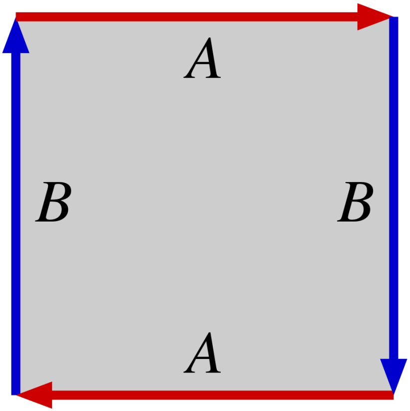
- One can often do better, *e.g.* S^n embeds into $\mathbb{R}^{n+1}$

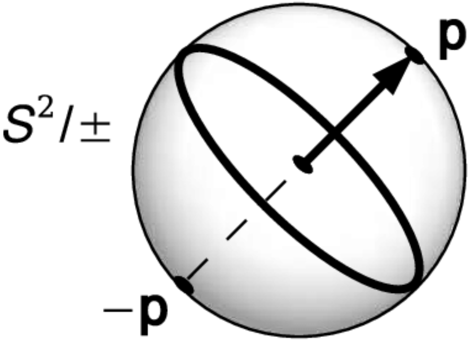
$\subset \mathbb{R}^3$

- If $n \neq 2^k$, then $\mathbb{R}^{2n-1}$ suffices for an embedding for compact connected n-mfds

Projective spaces are the party poopers for $n = 2^k$

We can't do better





- The real projective space $\mathbb{R}P^n$ of dim $n = 2^k$ can not be embedded in $\mathbb{R}^{2n-1}$
- The real projective space $\mathbb{R}P^n$ of dim $n = 2^k$ can not be immersed in $\mathbb{R}^{2n-2}$
- Thus, Whitney's theorems are optimal

Thank you for your attention!

I hope that was of some help.