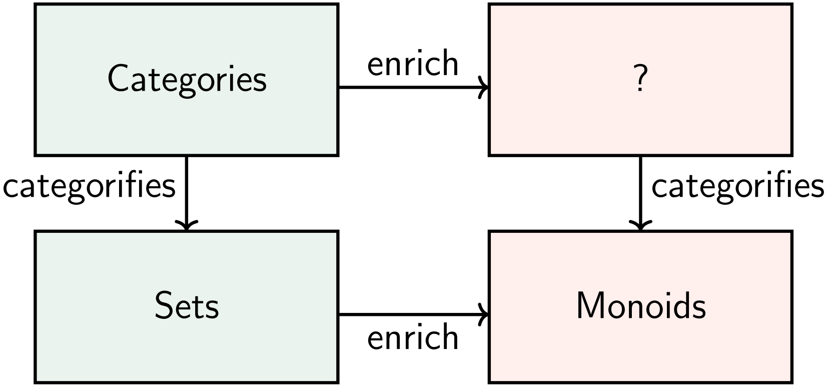
# What is...quantum topology - part 13?

Or: Monoidal categories 1 from Chapter 2

## Filling in the question mark



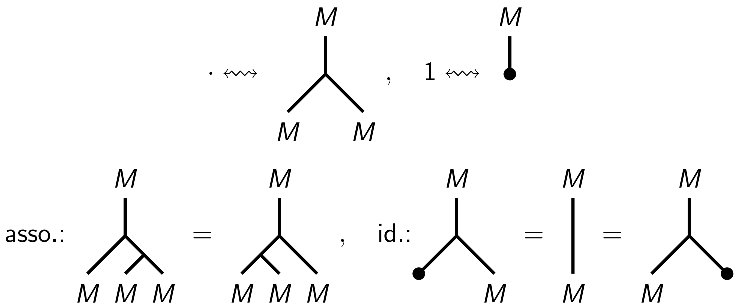

- Categories categorify sets
- Monoids are enriched sets
- What completes the square?

# What we do not want!

| Group-like structures | | | | | |
|---|---|---|---|---|---|
| | Totality$^{\alpha}$ | Associativity | Identity | Invertibility | Commutativity |
| Semigroupoid | Unneeded | Required | Unneeded | Unneeded | Unneeded |
| Small category | Unneeded | Required | Required | Unneeded | Unneeded |
| Groupoid | Unneeded | Required | Required | Required | Unneeded |
| Magma | Required | Unneeded | Unneeded | Unneeded | Unneeded |
| Quasigroup | Required | Unneeded | Unneeded | Required | Unneeded |
| Unital magma | Required | Unneeded | Required | Unneeded | Unneeded |
| Semigroup | Required | Required | Unneeded | Unneeded | Unneeded |
| Loop | Required | Unneeded | Required | Required | Unneeded |
| Inverse semigroup | Required | Required | Unneeded | Required | Unneeded |
| Monoid | Required | Required | Required | Unneeded | Unneeded |
| Commutative monoid | Required | Required | Required | Unneeded | Required |
| Group | Required | Required | Required | Required | Unneeded |
| Abelian group | Required | Required | Required | Required | Required |

- Monoids also appear *e.g.* via monads in categories
- Monads are not categorifications of monoids; just “similar in nature”
- A categorification should have two operations

# Monoids




A monoid $(M, \cdot)$ consists of

- A set $M$
- A multiplication $\cdot \colon M \times M \to M$ (write $ab = a \cdot b$)
- A unit $1 \in M$

such that

- $\cdot$ is associative $a(bc) = (ab)c$
- $\cdot$ is unital $1a = a = a1$

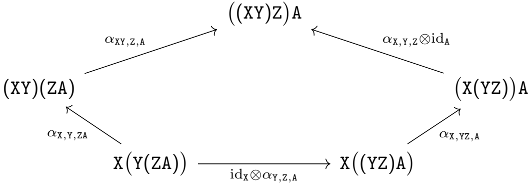
# For completeness: A formal definition

A **monoidal category** $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consists of

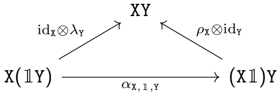
- A category $\mathcal{C}$
- A bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (write $XY = X \otimes Y$)
- A unit object $\mathbb{1} \in \mathcal{C}$
- Natural isomorphisms $\alpha_{X,Y,Z}\colon X(YZ) \to (XY)Z$, $\lambda_X\colon \mathbb{1}X \to X$, $\rho_X\colon X\mathbb{1} \to X$

**such that**

(a) the ⬠ *equality* holds, *i.e.* we have commuting diagrams



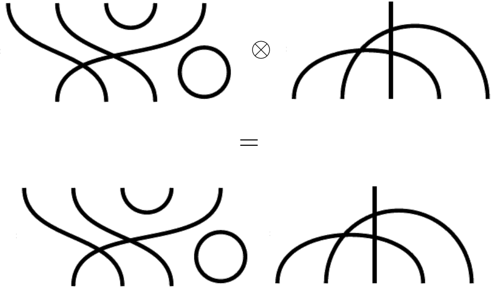

(b) the $\triangle$ *equality* holds, *i.e.* we have commuting diagrams



# Some examples

| Name | Objects | Arrows | $\otimes$ |
|---|---|---|---|
| Set | Sets | Maps | $\times$ |
| Cat | Categories | Functors | $\times$ |
| 1Cob | 0-manifolds | 1-manifolds | See below |
| nCob | (n-1)-manifolds | n-manifolds | Similarly as below |
| $\mathrm{Vec}_{\mathbb{K}}$ | $\mathbb{K}$-vector spaces | $\mathbb{K}$-linear maps | $\otimes$ |
| $\mathrm{Vec}_{\mathbb{K}}$ | $\mathbb{K}$-vector spaces | $\mathbb{K}$-linear maps | $\oplus$ |

Most diagrammatic categories are monoidal via juxtaposition:



**Thank you for your attention!**

I hope that was of some help.