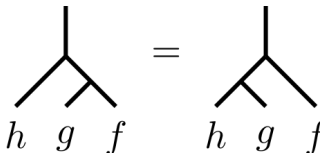


What is...quantum topology - part 14?

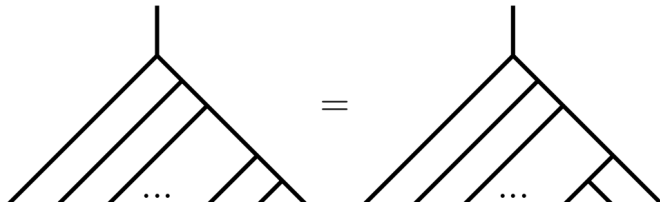
Or: Monoidal categories 2 from Chapter 2

A “wrong” and a “correct” definition

“Wrong” :

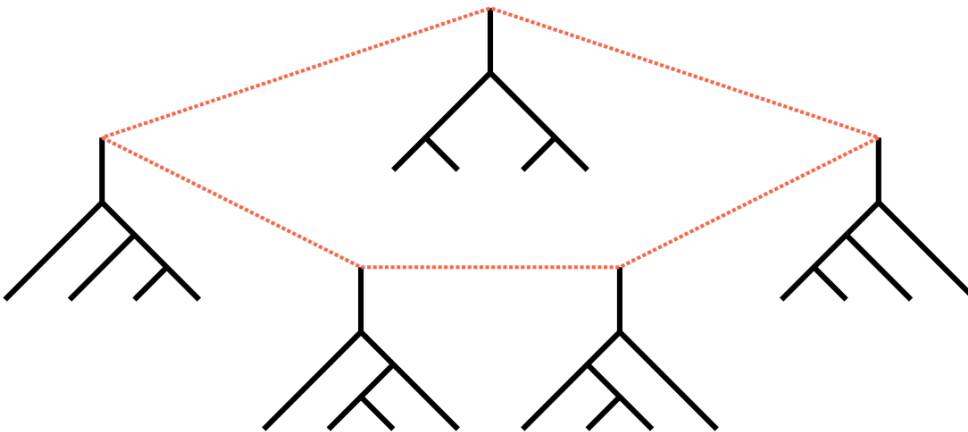
$$h(gf) = (hg)f$$


“Correct” :



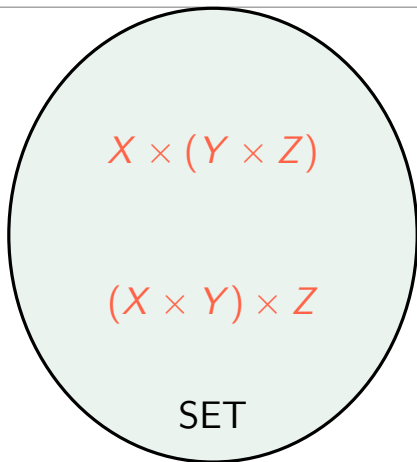
- ▶ (A) $h \cdot (g \cdot f) = (h \cdot g) \cdot f$
- ▶ (B) Same result regardless of how valid pairs of parentheses are inserted
- ▶ “Philosophically correct” Use (B) as the definition and show that $(A) \Leftrightarrow (B)$

Strategical interlude



-
- ▶ Define a monoid/group/... using $h \cdot (g \cdot f) = (h \cdot g) \cdot f$
 - ▶ Show that $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ implies all bracketings Coherence theorem
 - ▶ Forget parenthesis altogether Strictification

Why parenthesis in the first place?



-
- ▶ $X \times (Y \times Z) \neq (X \times Y) \times Z$ as sets Set theory is inflexible
 - ▶ In order to make SET with $\otimes = \times$ monoidal we need parenthesis
 - ▶ Use an equivalent category and avoid parenthesis Category theory is flexible

For completeness: A formal definition

A strict monoidal category $(C, \otimes, \mathbb{1})$ consists of

- ▶ A category C
- ▶ A bifunctor $\otimes: C \times C \rightarrow C$ (write $XY = X \otimes Y$)
- ▶ A unit object $\mathbb{1} \in C$

such that

(a) *associativity* holds, i.e.

$$X(YZ) = (XY)Z, \quad h \otimes (g \otimes f) = (h \otimes g) \otimes f$$

(b) the *identity law* holds, i.e.

$$\mathbb{1}X = X = X\mathbb{1}, \quad \text{id} \otimes f = f = f \otimes \text{id}$$

Theorem (strictification)

Every monoidal category is monoidally equivalent to a strict monoidal category

Some examples

Name	\otimes	Strict?	Strictification
Set	\times	No	$S(\text{Set})$
Cat	\times	No	$S(\text{Cat})$
1Cob	Juxtaposition	Yes	1Cob
nCob	Juxtaposition	Yes	nCob
$\text{Vec}_{\mathbb{K}}$	\otimes	No	$S(\text{Vec}_{\mathbb{K}})$
$\text{Vec}_{\mathbb{K}}$	\oplus	No	$S(\text{Vec}_{\mathbb{K}})$
$\text{End}(\mathcal{C})$	\circ	Yes	$\text{End}(\mathcal{C})$

The diagram illustrates the relationship between a category \mathcal{A} and its strictification $S(\mathcal{A})$. It shows a sequence of categories $\mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}''$ and their corresponding strictifications $S(\mathcal{A}) \rightarrow S(\mathcal{A}') \rightarrow S(\mathcal{A}'')$. The maps between the categories are $G: \mathcal{A} \rightarrow \mathcal{A}'$ and $G': \mathcal{A}' \rightarrow \mathcal{A}''$. The maps between the strictifications are $\alpha: S(\mathcal{A}) \rightarrow S(\mathcal{A}')$ and $\alpha': S(\mathcal{A}') \rightarrow S(\mathcal{A}'')$. The maps $F: \mathcal{A} \rightarrow S(\mathcal{A})$ and $F': \mathcal{A}' \rightarrow S(\mathcal{A}')$ are the natural isomorphisms from the categories to their strictifications. The maps $H: S(\mathcal{A}) \rightarrow \mathcal{A}'$ and $H': S(\mathcal{A}') \rightarrow \mathcal{A}''$ are the natural isomorphisms from the strictifications to the categories. The diagram shows that the strictification of a composition is the composition of the strictifications.

- Often the skeleton $S(\mathcal{C})$ is the strictification but **not** always
- In general the strictification is an **endofunctor category**

Thank you for your attention!

I hope that was of some help.