What is...quantum topology - part 28?

Or: Additive categories 1 from Chapter 6

Why addition enters the story

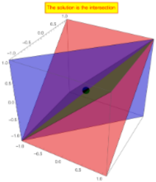
The art of solving linear equations

$$\begin{cases} 0x + 1/2y - 1/2z = 0 & \text{(Red)} \\ -1/2x + 0y + 1/2z = 0 & \text{(Green)} \\ 1/2x - 1/2y + 0z = 0 & \text{(Blue)} \end{cases}$$

or equivalently

$$\left(\begin{array}{ccc|c} 0 & 1/2 & -1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1/2 & 0 & 0 \end{array}\right)$$

$\rightsquigarrow$



The need for a machine to solve linear equations grew out of...

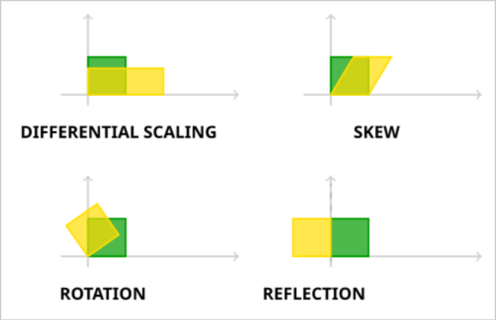
- ...Cartesian geometry
- ...the geometry of lines, planes and hyperplanes
- ...that solving other types of equations (polynomial or even worse) is very hard

Linear algebra provides algorithms to solve linear problems

FIGURE 12. The essence of linear algebra.

- Braiding etc. gave us a way to read diagrams
- But invariants should also be computable
- So we need to add the categorical version of linear algebra

Adding morphisms



Examples of linear transformations used in computer graphics

- Between two vector spaces, maps themselves form a vector space
- This means we can add maps, subtract maps and use a zero map
- An additive category remembers exactly this bookkeeping of arrows

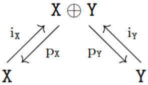
Zero objects and direct sums

Finally, we consider the pair category (see Definition 1B.12) $\mathbf{Vec}_{\Bbbk} \times \mathbf{Vec}_{\Bbbk}$ and we have a bifunctor

$$\oplus\colon \mathbf{Vec}_{\Bbbk} \times \mathbf{Vec}_{\Bbbk} \to \mathbf{Vec}_{\Bbbk}, \quad \oplus\big((\mathtt{X},\mathtt{Y})\big) = \mathtt{X} \oplus \mathtt{Y}, \oplus\big((f,g)\big) = f \oplus g,$$

called the ***direct sum***, using again abbreviations of the form $\mathtt{X} \oplus \mathtt{Y}$ instead of $\oplus\big((\mathtt{X},\mathtt{Y})\big)$. We note that the object $\mathtt{X} \oplus \mathtt{Y}$ has a universal-type property, namely: there exist morphisms $i_{\mathtt{X}}, i_{\mathtt{Y}}, p_{\mathtt{X}}, p_{\mathtt{Y}} \in \mathbf{Vec}_{\Bbbk}$ such that

(6B-3)



$$, \quad p_{\mathtt{X}} i_{\mathtt{X}} = \mathrm{id}_{\mathtt{X}}, \quad p_{\mathtt{Y}} i_{\mathtt{Y}} = \mathrm{id}_{\mathtt{Y}}, \quad i_{\mathtt{X}} p_{\mathtt{X}} + i_{\mathtt{Y}} p_{\mathtt{Y}} = \mathrm{id}_{\mathtt{X} \oplus \mathtt{Y}}.$$

The two morphisms $i_{\mathtt{X}}$ and $i_{\mathtt{Y}}$ are called ***inclusions***, the other two $p_{\mathtt{X}}$ and $p_{\mathtt{Y}}$ ***projections*** (of $\mathtt{X}$ and $\mathtt{Y}$, respectively).

- The zero object is the categorical version of nothing to see here
- A direct sum puts two objects side by side without forcing them to interact
- Inclusions and projections are the handles for moving in and out; the abstract definition is above

Matrices already know this

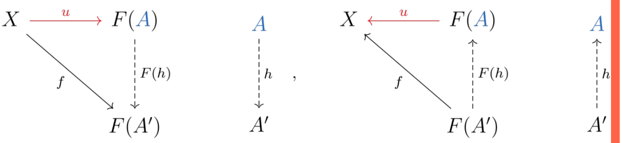
$$A \oplus B = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{p1} & \cdots & b_{pq} \end{bmatrix}$$

- Matrix addition is the most down-to-earth model example
- Block matrices are direct sums wearing a very visible costume
- The abstract definitions only package what matrices have done all along

The universal-property mindset

For completeness: A formal definition

- $F\colon C \to D$, $X \in D$, a universal morphism from X to F is a pair $(A, u\colon X \to F(X))$ such that $\exists! h$ making the left diagram below commute
- $F\colon C \to D$, $X \in D$, a universal morphism from F to X is a pair $(A, u\colon F(X) \to X)$ such that $\exists! h$ making the right diagram below commute



- These might not exists
- If they exist, then they are unique up to unique isomorphism

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- We often define objects by what they do, not by how they are built
- This is why direct sums are controlled by maps in and maps out
- Reward = uniqueness up to unique isomorphism = as canonical as it gets

Thank you for your attention!