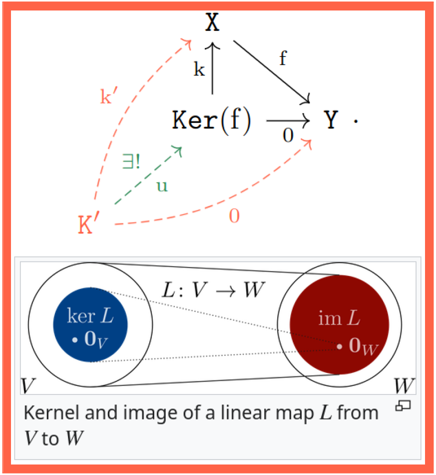
What is...quantum topology - part 30?

Or: Additive categories 3 from Chapter 6

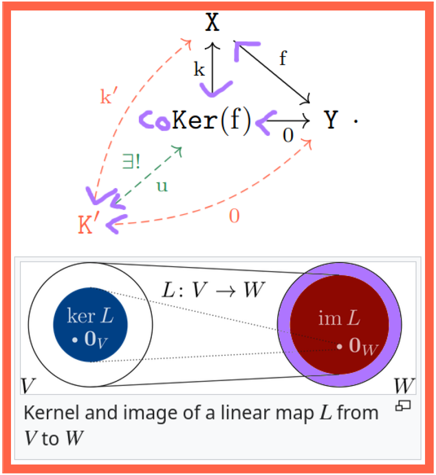
What gets sent to zero



Kernel and image of a linear map L from V to W

- For $f : X \to Y$, the kernel remembers the part of X sent to zero
- Every map killed by f factors uniquely through the kernel
- Thus being killed by f makes sense categorically, even without elements

Cokernels turn hits into quotients



Kernel and image of a linear map L from V to W

- The cokernel is the dual construction: it kills what f already hits
- For vector spaces, $\mathrm{Coker}(f) = Y/\mathrm{Im}(f)$: quotient out the image
- Kernels behave like subobjects; cokernels like quotient objects

Images are what survives

coker

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \mathrm{im} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$2a - b$$

- The image is the part of the target actually reached by f
- Categorically, image means kernel of the cokernel
- With an epic–monic factorization, image and coimage give the same middle

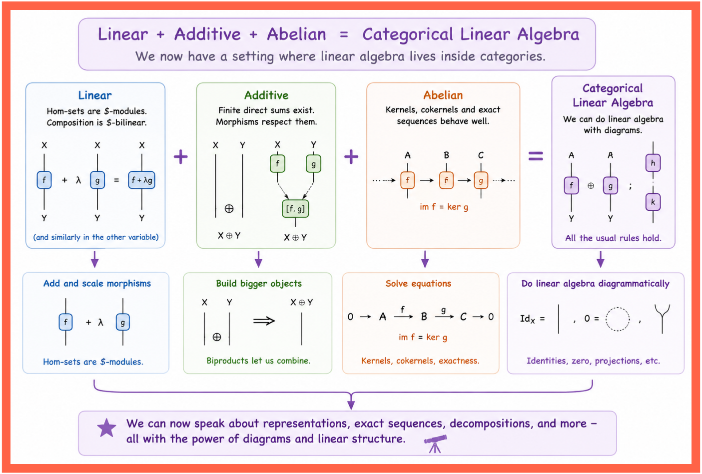
Every map wants a middle

$$\begin{bmatrix} \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of a rectangular matrix in row echelon form

- An epic–monic factorization writes $X \twoheadrightarrow I \hookrightarrow Y$
- For matrices, row reduction finds I; over a field its dimension is the rank
- An abelian category is additive and every morphism has such a factorization

Why abelian matters



- Vector spaces and module categories are abelian
- Freyd–Mitchell puts every abelian category inside some $\mathrm{Mod}(A)$
- Tensor categories add $\otimes$: representation theory meets quantum topology

Thank you for your attention!