What are...characters?

Or: Polynomials!

My wish list

To each rep I want an associated numerical invariant

Numerical invariant = something like a number

The invariant should behave nicely wrt operations on reps

The invariant should determine the rep

- The idea of invariants is ubiquitous in mathematics/the sciences
- So let's apply it in rep theory!
- However, the last point sounds impossible

Traces!

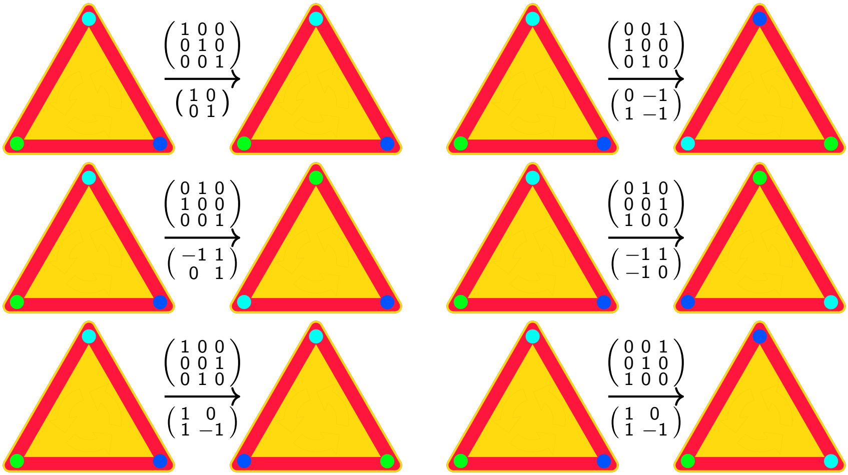
$$\begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{bmatrix}$$

$$\begin{aligned} \mathrm{tr}(\mathbf{A} + \mathbf{B}) &= \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B}) \\ \mathrm{tr}(c\mathbf{A}) &= c\,\mathrm{tr}(\mathbf{A}) \\ \mathrm{tr}(\mathbf{AB}) &= \mathrm{tr}(\mathbf{BA}) \end{aligned}$$

$$\mathrm{tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B})$$

- Traces are invariant under base change
- Traces are like polynomials

Permutation = trivial ⊕ standard



- Traces are 3|1|2, 1|1|0, 1|1|0, 0|1|-1, 0|1|-1, 1|1|0 for perm—stand—triv
- Trace(perm)=Trace(triv)+Trace(stand) ⇔? $V_{perm} \cong L_{triv} \oplus L_{stand}$

For completeness: A formal definition

ϕ a G-representation on a $\mathbb{K}$-vector space V; the character χ_ϕ is the map

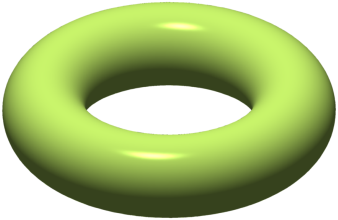
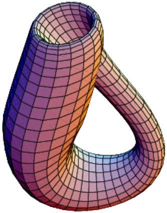
$$\chi_\phi \colon G \to \mathbb{K}, g \mapsto \mathrm{tr}(\phi_g)$$

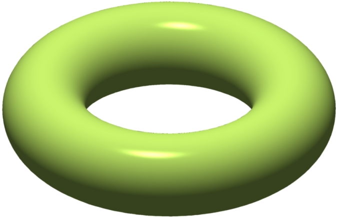
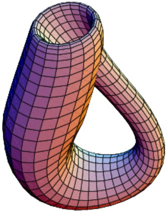
Properties of characters

- $\phi = \psi \Rightarrow \chi_\phi = \chi_\psi$ Invariance
- $\chi_{\phi\oplus\psi} = \chi_\phi + \chi_\psi, \quad \chi_{\phi\otimes\psi} = \chi_\phi \cdot \chi_\psi$ Polynomial properties
- Over $\mathbb{C}$: $\phi = \psi \Leftarrow \chi_\phi = \chi_\psi$ Character determines rep
- Characters are constant on conjugacy classes

	(1)	(12)	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	−1	1
χ_{stand}	2	0	−1

The finite group miracle

Euler char () $=$ Euler char ()

 $\not\simeq$ 

- Over $\mathbb{C}$: $\phi = \psi \Leftrightarrow \chi_\phi = \chi_\psi$ Character determines rep
- Analogs in other fields are often very wrong

Thank you for your attention!

I hope that was of some help.