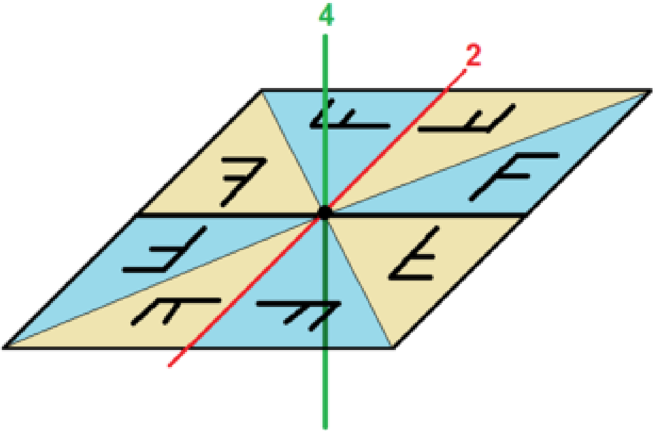
What is...the regular representation?

Or: Action on itself

Dihedral groups

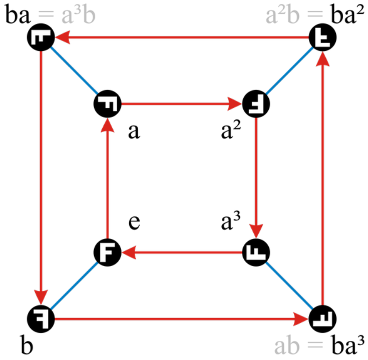
D_4 acts on



- Dihedral groups $D_n = \langle a, b \rangle$ are the symmetry groups of $n$gons
- Slight problem D_n has 2n elements, but $n$gons gives a vector space of dim n

Action on itself

D_4 acts on D_4



- Every group can act on itself
- The underlying geometric object can be thought of as the Cayley graph
- The linear version is called the regular representation

D_4 **acts on** $\mathbb{C}[D_4]$

$$a \leftrightsquigarrow \begin{array}{c} \\ id \\ a \\ a^2 \\ a^3 \\ b \\ ab \\ a^2b \\ a^3b \end{array} \begin{array}{c} \begin{array}{cccccccc} id & a & a^2 & a^3 & b & ab & a^2b & a^3b \end{array} \\ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$id, b \leftrightsquigarrow$ Not displayed - no space

- The trace of id on $\mathbb{C}[D_4]$ is 8
- The traces of a and b on $\mathbb{C}[D_4]$ are zero

For completeness: A formal definition

The regular rep R is the $\mathbb{K}$-vector space $\mathbb{K}[G]$ with action by left multiplication

- Strictly speaking this should be called left regular rep
- The regular rep makes sense for any group
- Its dimension is always $|G|$

Cool facts (easy to show):

- We have the character table

R:

Class	1	2	3	...
Size	1	C_2	C_3	...
ξ_R	$\|G\|$	0	0	...

- For $\mathbb{K} = \mathbb{C}$ we have

$$R \cong L_1^{\oplus \dim L_1} \oplus \ldots \oplus L_r^{\oplus \dim L_r}$$

$$\xi_R = \dim L_1 \cdot \chi_1 + \ldots + \dim L_r \cdot \chi_r$$

and all simple reps L_k appear

More semisimple miracles

```
G := DihedralGroup(4);
CT := CharacterTable(G);
CT;
```

```
------------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
------------------------
p  =  2   1  1  1  1  2
------------------------
X.1    +   1  1  1  1  1
X.2    +   1  1 -1  1 -1
X.3    +   1  1  1 -1 -1
X.4    +   1  1 -1 -1  1
X.5    +   2 -2  0  0  0
```

- The regular representation is

$$R \cong L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_5 \oplus L_5$$

L_5 appears twice

- Weighted sum of columns $= |G|$ respectively $= 0$ Numerical miracle

Thank you for your attention!

I hope that was of some help.