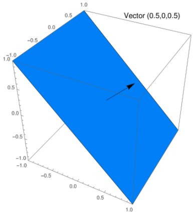
What is...the dual group?

Or: Groups of characters

Dual vector spaces and dual groups

$v = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \leftrightsquigarrow$



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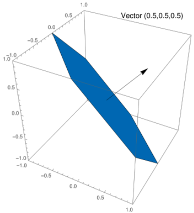
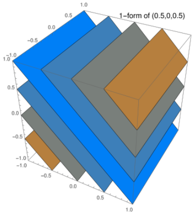


$\langle v, \_ \rangle \leftrightsquigarrow$



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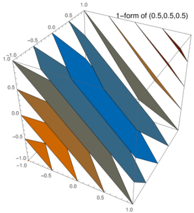

- Dual vector space = forms $\langle v, - \rangle \colon V \to \mathbb{C}$ on vector spaces
- Characters = maps $\chi \colon G \to \mathbb{C}$ on groups
- Dual group = group of characters?

Characters of cyclic groups

```
G := AbelianGroup([7]);
CT := CharacterTable(G);
CT;
```

```
-----------------------------------------
Class |   1    2    3    4    5    6    7
Size  |   1    1    1    1    1    1    1
Order |   1    7    7    7    7    7    7
-----------------------------------------
p  =  7   1    1    1    1    1    1    1
-----------------------------------------
X.1   +   1    1    1    1    1    1    1
X.2   0   1   Z1 Z1#2 Z1#3 Z1#4 Z1#5 Z1#6
X.3   0   1 Z1#4   Z1 Z1#5 Z1#2 Z1#6 Z1#3
X.4   0   1 Z1#3 Z1#6 Z1#2 Z1#5   Z1 Z1#4
X.5   0   1 Z1#6 Z1#5 Z1#4 Z1#3 Z1#2   Z1
X.6   0   1 Z1#5 Z1#3   Z1 Z1#6 Z1#4 Z1#2
X.7   0   1 Z1#2 Z1#4 Z1#6   Z1 Z1#3 Z1#5


Explanation of Character Value Symbols
--------------------------------------

# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

Z1     = (CyclotomicField(7: Sparse := true)) ! [ RationalField() | 0, 0, 0, 1,
0, 0 ]
```

- Cyclic groups $C_n \cong (\mathbb{Z}/n\mathbb{Z}, +)$ are the rotational symmetry groups of $n$gons
- Theorem The simple characters of C_n are given by nth roots of unity
- Group structure $\Leftrightarrow$ multiplication of characters

Characters of finite abelian groups

```
G := AbelianGroup([2,3]);
CT := CharacterTable(G);
CT;
```

```
--------------------------------
Class |   1  2    3    4    5    6
Size  |   1  1    1    1    1    1
Order |   1  2    3    3    6    6
--------------------------------
p  =  2   1  1    4    3    3    4
p  =  3   1  2    1    1    2    2
--------------------------------
X.1   +   1  1    1    1    1    1
X.2   +   1 -1    1    1   -1   -1
X.3   0   1  1    J -1-J -1-J    J
X.4   0   1 -1 -1-J    J   -J  1+J
X.5   0   1 -1    J -1-J  1+J   -J
X.6   0   1  1 -1-J    J    J -1-J


Explanation of Character Value Symbols
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J = RootOfUnity(3)
```

- **Theorem** All finite abelian groups are products of cyclic groups
- **Theorem** The simple characters of finite abelian groups are products of the simple characters of cyclic groups
- **Group structure ⇔ multiplication of characters**

For completeness: A formal definition

For a locally compact abelian (lca) group G, the dual group is

$$\widehat{G} = \hom_{\mathsf{topGROUP}}(G, \mathbb{C}^\times)$$

- Example of lca groups: finite abelian groups, $\mathbb{Z}$, $\mathbb{R}$, S^1
- For finite abelian groups we take the discrete topology, thus can forget about it
- Careful that $\mathbb{C}^\times$ is sometimes replaced by a different group
- Can be defined for non-commutative groups, but is not as useful

Facts (easy to show):

- Equipped with pointwise multiplication, $\widehat{G}$ is an abelian group
- $\widehat{G}$ is the group of characters
- $G \cong \widehat{\widehat{G}}$ canonically
- $\mathbb{Z}/n\mathbb{Z} \cong \widehat{\mathbb{Z}/n\mathbb{Z}}$

Characters = group

```
G := AbelianGroup([7]);
CT := CharacterTable(G);
CT;
```

```
----------------------------------------
Class |   1    2    3    4    5    6    7
Size  |   1    1    1    1    1    1    1
Order |   1    7    7    7    7    7    7
----------------------------------------
p  =  7   1    1    1    1    1    1    1
----------------------------------------
X.1   +   1    1    1    1    1    1    1
X.2   0   1   Z1 Z1#2 Z1#3 Z1#4 Z1#5 Z1#6
X.3   0   1 Z1#4   Z1 Z1#5 Z1#2 Z1#6 Z1#3
X.4   0   1 Z1#3 Z1#6 Z1#2 Z1#5   Z1 Z1#4
X.5   0   1 Z1#6 Z1#5 Z1#4 Z1#3 Z1#2   Z1
X.6   0   1 Z1#5 Z1#3   Z1 Z1#6 Z1#4 Z1#2
X.7   0   1 Z1#2 Z1#4 Z1#6   Z1 Z1#3 Z1#5


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# denotes algebraic conjugation, that is,
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```

- $\chi_1 \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$, $1 \mapsto \exp(2\pi i/n)$
- $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\cong} \widehat{\mathbb{Z}/n\mathbb{Z}}$ via $1 \mapsto \chi_1$
- Addition $\mapsto$ multiplication

Thank you for your attention!

I hope that was of some help.