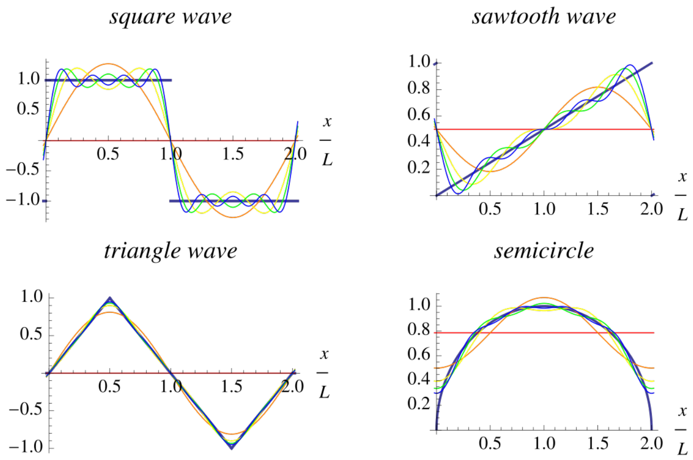
What is…finite Fourier analysis?

Or: Fourier and finite groups

Fourier series



- $f : \mathbb{R} \to \mathbb{C}$ period function $f(x+n) = f(x)$
- Fourier series $f(x) = \sum_{k=-n}^{n} c_k \cdot \exp(2\pi i x \cdot k/n)$
- The Fourier transform encodes this information as a function

Fourier transform

Fourier transform integral

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx, \quad \forall\, \xi \in \mathbb{R}.$$

Fourier inversion integral

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi, \quad \forall\, x \in \mathbb{R}.$$

- $f : \mathbb{R} \to \mathbb{C}$ (reasonably nice) starting function
- $\hat{f} : \mathbb{R} \to \mathbb{C}$ Fourier transform
- $f = \hat{\hat{f}}$ Dual pair

Two perspectives on the dual group

```
G := AbelianGroup([7]);
CT := CharacterTable(G);
CT;
```

```
-----------------------------------------
Class |   1    2    3    4    5    6    7
Size  |   1    1    1    1    1    1    1
Order |   1    7    7    7    7    7    7
-----------------------------------------
p  =  7   1    1    1    1    1    1    1
-----------------------------------------
X.1   +   1    1    1    1    1    1    1
X.2   0   1   Z1 Z1#2 Z1#3 Z1#4 Z1#5 Z1#6
X.3   0   1 Z1#4   Z1 Z1#5 Z1#2 Z1#6 Z1#3
X.4   0   1 Z1#3 Z1#6 Z1#2 Z1#5   Z1 Z1#4
X.5   0   1 Z1#6 Z1#5 Z1#4 Z1#3 Z1#2   Z1
X.6   0   1 Z1#5 Z1#3   Z1 Z1#6 Z1#4 Z1#2
X.7   0   1 Z1#2 Z1#4 Z1#6   Z1 Z1#3 Z1#5


Explanation of Character Value Symbols
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# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

Z1     = (CyclotomicField(7: Sparse := true)) ! [ RationalField() | 0, 0, 0, 1,
0, 0 ]
```

- $\widehat{\mathbb{Z}/n\mathbb{Z}}$ = (simple characters of $\mathbb{Z}/n\mathbb{Z}$) = (nth roots of unity)
- $\widehat{\mathbb{Z}/n\mathbb{Z}}$ = (functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$) = ($n$-periodic functions $\mathbb{Z} \to \mathbb{C}$)
- What is Fourier analysis in this setting?

For completeness: A formal definition

Let $f\colon G \to \mathbb{C}$ be a function on a finite abelian group G

Fourier transform: $\hat{f}\colon \hat{G} \to \mathbb{C}$ with $\hat{f}(\chi) = \sum_{g \in G} f(g)\overline{\chi(g)}$

$\hat{\hat{f}} = f$ and the Fourier transform is invertible with inverse

$$f(x) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(x)$$

Let $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ be a periodic function

Fourier transform: $\hat{f}\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ with $\hat{f}(x) = \sum_{k=0}^{n-1} f(k) \exp(-2\pi i x \cdot k/n)$

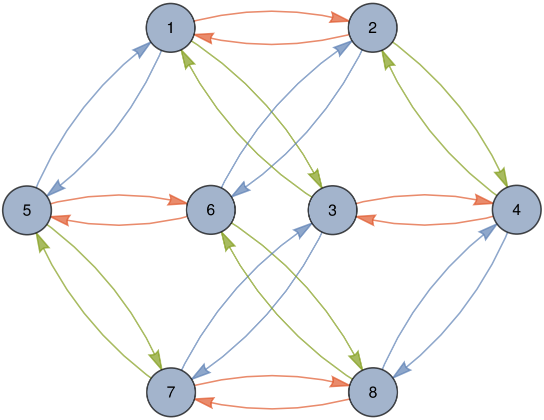
$\hat{\hat{f}} = f$ and the Fourier transform is invertible with inverse

$$f(x) = \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}(k) \exp(2\pi i x \cdot k/n)$$

For $G = \mathbb{Z}/n\mathbb{Z}$ the above are the same via $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$

A surprising application

$(\mathbb{Z}/2\mathbb{Z})^3$:



Eigenvalues: $3, -3, 1, 1, 1, -1, -1, -1$

- The eigenvalues of Cayley graph for finite abelian groups have real eigenvalues
- This can be proven using finite Fourier analysis

Thank you for your attention!

I hope that was of some help.