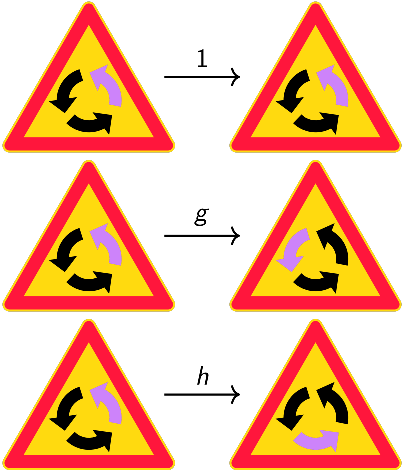
# What are...representations?

Or: Matrices are cool!

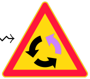
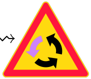
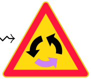
## From nonlinear to linear

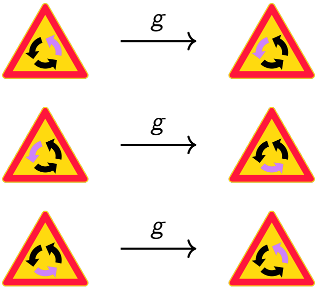
$\mathbb{Z}/3\mathbb{Z}$ acts on




- A group is a nonlinear object; a representation should be a linear object
- Why? Because “linear=easy”

## Vectors and matrices

$\mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{C}\Big\{(1,0,0) \leftrightsquigarrow$  $,(0,1,0) \leftrightsquigarrow$  $,(0,0,1) \leftrightsquigarrow$  $\Big\}$




$$g \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- A representation replaces geometric objects by vectors
- A representation replaces actions by matrices

# Groups in matrices

| $\cdot$ | $1$ | $g$ | $h$ |
|---|---|---|---|
| $1$ | $1$ | $g$ | $h$ |
| $g$ | $g$ | $h$ | $1$ |
| $h$ | $h$ | $1$ | $g$ |

*e.g.* $gh = 1$

| $\cdot$ | $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$ | $\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$ |
|---|---|---|---|
| $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$ | $\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$ |
| $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$ | $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$ | $\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$ | $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ |
| $\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$ | $\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$ | $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$ |

*e.g.* $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$

- The assignment

$$1 \mapsto \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, g \mapsto \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}, h \mapsto \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$$

  is a representation of $\mathbb{Z}/3\mathbb{Z}$

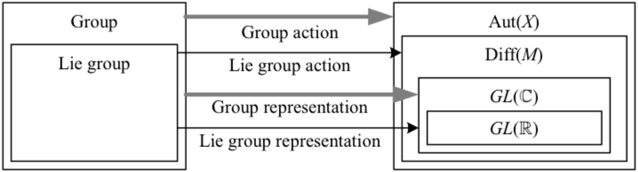
- This transfers questions about $\mathbb{Z}/3\mathbb{Z}$ to linear algebra

## For completeness: A formal definition

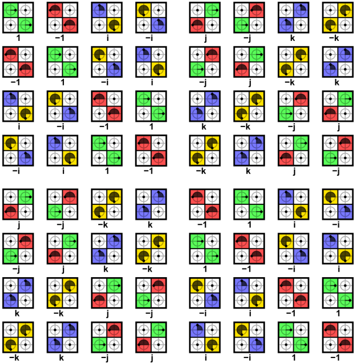
A representation of a group $G$ on a $\mathbb{K}$-vector space $V$ is a group homomorphism

$$\phi\colon G \to \mathrm{Aut}(V) = \mathrm{GL}(V)$$

- $\phi_g = \phi(g)$ is a matrix!
- There is always the trivial representation $g \mapsto (\,1\,)$
- The ground field plays a crucial role and $G$-representation might vary with $\mathbb{K}$
- Representations work more generally, *e.g.* for Lie groups



# Efficiency!?




| × | e | $\bar{e}$ | i | $\bar{i}$ | j | $\bar{j}$ | k | $\bar{k}$ |
|---|---|---|---|---|---|---|---|---|
| **e** | e | $\bar{e}$ | i | $\bar{i}$ | j | $\bar{j}$ | k | $\bar{k}$ |
| **$\bar{e}$** | $\bar{e}$ | e | $\bar{i}$ | i | $\bar{j}$ | j | $\bar{k}$ | k |
| **i** | i | $\bar{i}$ | $\bar{e}$ | e | k | $\bar{k}$ | $\bar{j}$ | j |
| **$\bar{i}$** | $\bar{i}$ | i | e | $\bar{e}$ | $\bar{k}$ | k | j | $\bar{j}$ |
| **j** | j | $\bar{j}$ | $\bar{k}$ | k | $\bar{e}$ | e | i | $\bar{i}$ |
| **$\bar{j}$** | $\bar{j}$ | j | k | $\bar{k}$ | e | $\bar{e}$ | $\bar{i}$ | i |
| **k** | k | $\bar{k}$ | j | $\bar{j}$ | $\bar{i}$ | i | $\bar{e}$ | e |
| **$\bar{k}$** | $\bar{k}$ | k | $\bar{j}$ | j | i | $\bar{i}$ | e | $\bar{e}$ |

- Groups might have very efficient representations
- Example The quaternion group of order 8 can be faithfully represented on $\mathbb{C}^2$:

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad k \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\bar{e} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \bar{i} \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \bar{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \bar{k} \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

**Thank you for your attention!**

I hope that was of some help.