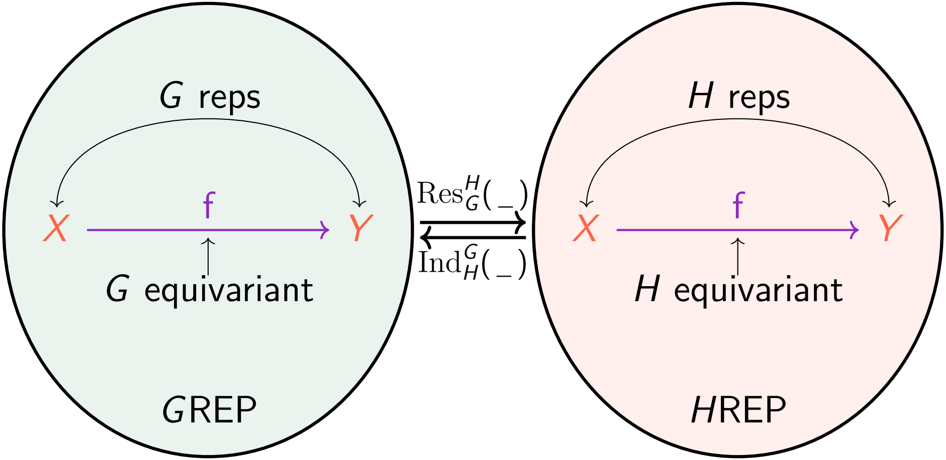
What is...Frobenius reciprocity?

Or: Back-and-forth

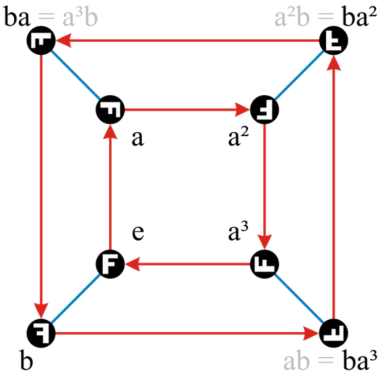
From G to H to G



- Restriction $\mathrm{Res}_G^H(\_)$ goes from G reps to H reps
- Induction $\mathrm{Ind}_H^G(\_)$ goes from H reps to G reps
- Frobenius reciprocity explains the relation between them

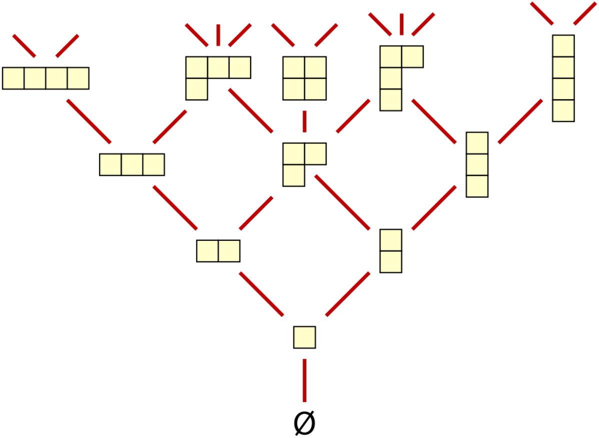
Example (regular rep)

D_4 acts on D_4



- $H = 1 \subset G$
- $\langle \chi_{\mathrm{Ind}_H^G(1)}, \chi_1 \rangle = \langle \chi_1, \chi_{\mathrm{Res}_G^H(1)} \rangle = 1$
- Why? Because $\mathrm{Res}_G^H(1) \cong 1$ and $\mathrm{Ind}_H^G(1)$ is the regular rep

Ind-res for symmetric groups



- $H = S_2 \subset G = S_3$, $[G : H] = 3$
- $\langle \chi_{\mathrm{Ind}_H^G(L_{triv})}, \chi_{L_{triv}} \rangle = \langle \chi_{L_{triv}}, \chi_{\mathrm{Res}_G^1(L_{triv})} \rangle = 1$
- But $\mathrm{Ind}_H^G(L_{triv})$ is of dim 3, so there is another summand!

For completeness: A formal statement

Frobenius reciprocity:

$$\mathrm{Hom}_{G\mathsf{REP}}\big(\mathrm{Ind}_H^G(\phi), \psi\big) \cong \mathrm{Hom}_{H\mathsf{REP}}\big(\phi, \mathrm{Res}_G^H(\psi)\big)$$

$$\mathrm{Hom}_{G\mathsf{REP}}\big(\psi, \mathrm{Ind}_H^G(\phi)\big) \cong \mathrm{Hom}_{H\mathsf{REP}}\big(\mathrm{Res}_G^H(\psi), \phi\big)$$

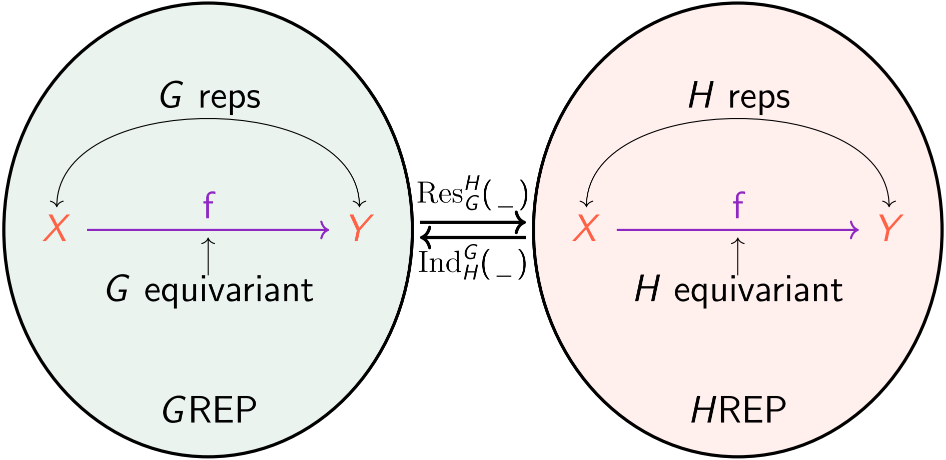
- $H \subset G$ is a subgroup
- ϕ is an H rep, ψ a G rep, ground field $\mathbb{C}$
- Character version

$$\langle \chi_{\mathrm{Ind}_H^G(\phi)}, \chi_\psi \rangle = \langle \chi_\phi, \chi_{\mathrm{Res}_G^H(\psi)} \rangle$$

$$\langle \chi_\psi, \chi_{\mathrm{Ind}_H^G(\phi)} \rangle = \langle \chi_{\mathrm{Res}_G^H(\psi)}, \chi_\phi \rangle$$

Hermitian adjoints

From G to H to G - revisited



- Frobenius reciprocity says that $\left(\mathrm{Ind}^G_H(\_), \mathrm{Res}^H_G(\_)\right)$ is an adjoint pair
- Frobenius reciprocity also says that $\left(\mathrm{Res}^H_G(\_), \mathrm{Ind}^G_H(\_)\right)$ is an adjoint pair
- The first is true in general, the second a finite group miracle

Thank you for your attention!

I hope that was of some help.