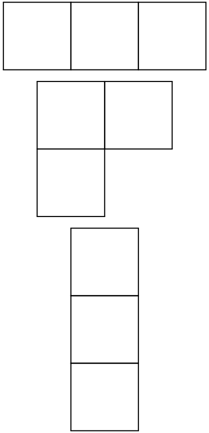
What is...Frobenius' character formula?

Or: Representations of symmetric groups, part 3

Searching for a character formula

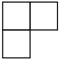
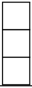
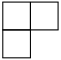
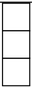


$\xleftrightarrow{XYZ}$

```
------------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
------------------
p  =  2   1  1  3
p  =  3   1  2  1
------------------
X.1    +   1  1  1
X.2    +   1 -1  1
X.3    +   2  0 -1
```

- Simple reps of S_n $\leftrightsquigarrow$ Young diagrams with n boxes
- Character of simple reps of S_n $\leftrightsquigarrow$ XYZ Young diagrams with n boxes
- Frobenius' character formula fills XYZ

Two Young diagrams

			
	1	1	1
	1	-1	1
	2	0	-1

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
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p  =  2   1  1  3
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X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```
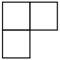

- We also know that conjugacy classes are index by Young diagrams
- Goal Associate a number to any pair of Young diagrams

Enter, Frobenius' counting

 $\leftrightsquigarrow x_1^{\boxed{3}} x_2^{\boxed{1}}$ $\quad l_1 = 2 + 2 - 1 = \boxed{3}, l_2 = 1 + 2 - 2 = \boxed{1}$

 $\leftrightsquigarrow (x_1 + x_2)^{\boxed{3}} \quad i_1 = \boxed{3}$

$$(x_1 - x_2)(x_1 + x_2)^4 = x_1^4 + \boxed{2}\, x_1^{\boxed{3}} x_2^{\boxed{1}} - 2x_1x_2^3 - x_2^4$$

- Given $\lambda = (\lambda_1, ..., \lambda_k)$ and μ Young diagram
- $l_s = \lambda_s + k - s =$ length of rows + total length - row number
- $i_s = \#$ columns of length s
- k variables x_1 to x_k
- From μ get a polynomial

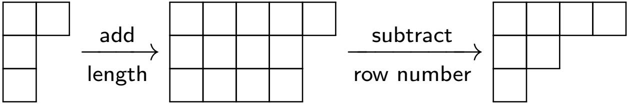
$$P = \prod_{i<j}(x_i - x_j) \prod_j (x_1^j + ... + x_k^j)^{i_j}$$

- Look for the coefficient of $x_1^{l_1} ... x_k^{l_k}$ in P

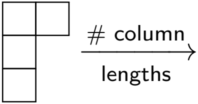
For completeness: A formal statement

The character of χ_λ on μ can be computed by Frobenius' counting

Another example



$\rightsquigarrow x_1^4 x_2^2 x_3$



$i_1 = 1, i_3 = 1$

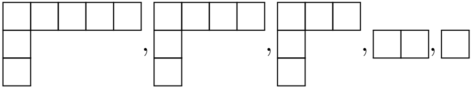
$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_2)(x_1 + x_2 + x_3)(x_1^3 + x_2^3 + x_3^3) =$$

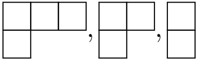
$$\ldots \boxed{0}\, x_1^4 x_2^2 x_3 + \ldots$$

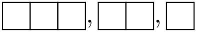
$\Rightarrow$ character value is 0

Hook length formula

$$\begin{array}{|c|c|c|c|c|}\hline 7 & 6 & 5 & 2 & 1 \\ \hline 4 & 3 & 2 \\ \cline{1-3} 3 & 2 & 1 \\ \cline{1-3}\end{array}$$

first row:  $\rightsquigarrow$ $7, 6, 5, 2, 1$

second row:  $\rightsquigarrow$ $4, 3, 2$

second row:  $\rightsquigarrow$ $3, 2, 1$

$$\text{dim is } \frac{11!}{7 \cdot 6 \cdot 5 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1} = 660$$

- Hook length formula = product of the hook length gives the dim of the simples
- Frobenius' character formula $\Rightarrow$ hook length formula

Thank you for your attention!

I hope that was of some help.