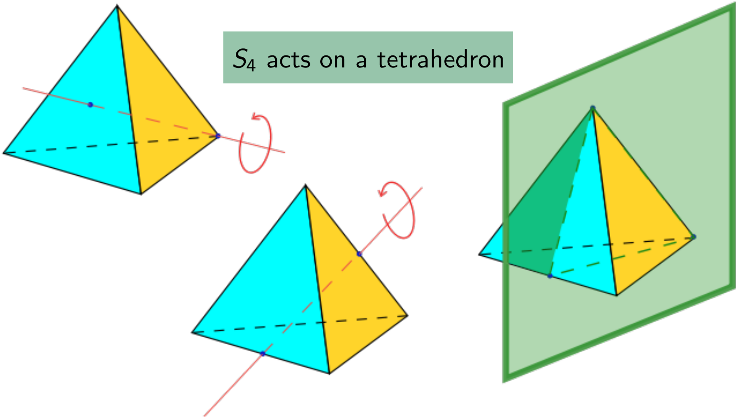
What are...modules?

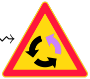
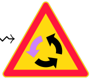
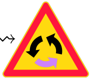
Or: Linear symmetries

Symmetries are actions



- Symmetry = a property that remains unchanged under operations
- In this sense "symmetries=actions"
- Slogan: Modules=linear actions=symmetries of vector spaces

Same slide as in the previous video – almost ;-)

$\mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{C}\Big\{(1,0,0) \leftrightsquigarrow$  $, (0,1,0) \leftrightsquigarrow$  $, (0,0,1) \leftrightsquigarrow$  $\Big\}$

 $\xrightarrow{g}$ 

 $\xrightarrow{g}$ 

 $\xrightarrow{g}$ 

$$g \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- A module replaces geometric objects by vectors
- In a module groups act by matrices

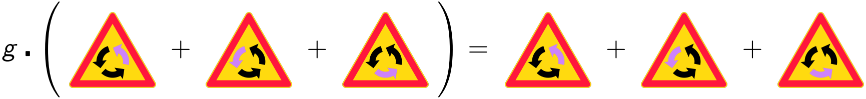
Linear algebra!

$$v = (1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$$

$$\leftrightsquigarrow$$

$$g \centerdot v = v$$

$$\leftrightsquigarrow$$



- In a module one can take linear combinations
- Game changer The formal linear combination of pictures is an eigenvector

For completeness: A formal definition

A (left) module of G is a $\mathbb{K}$-vector space with an action $\centerdot: G \times V \to V$ such that:

- $1 \centerdot v = v$ Unitality
- $h \centerdot (g \centerdot v) = hg \centerdot v$ Associativity
- $g \centerdot (\lambda v + \mu w) = \lambda (g \centerdot v) + \mu (g \centerdot w)$ Linearity

There are "nice" bijections (for fixed $\mathbb{K}$)

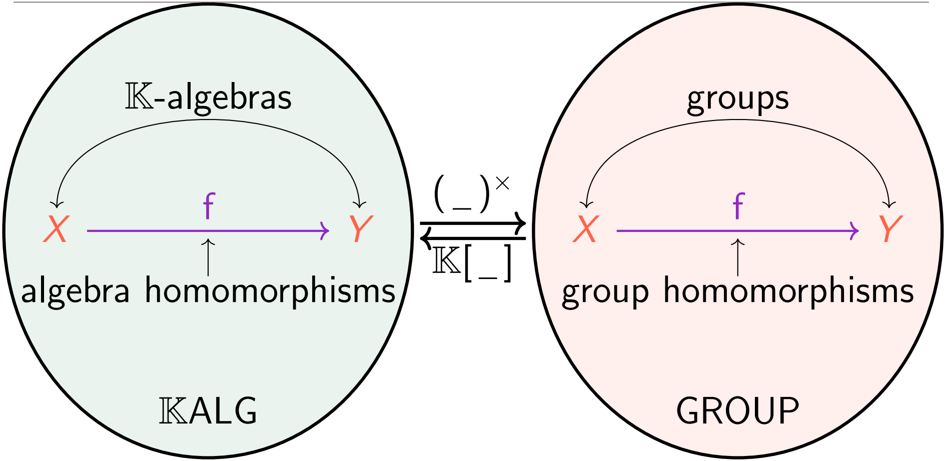
G-reps $\xleftrightarrow{1:1}$ G-modules $\xleftrightarrow{1:1}$ $\mathbb{K}[G]$-modules

Representations and modules are two sides of the same coin

Reps  Modules

Difference: representations highlight the group, modules the action

What does “nice” mean?



- Equivalence of categories G-REP $\cong \mathbb{K}[G]$-MOD (for fixed $\mathbb{K}$)
- $(\mathbb{K}[\_], (\_)^{\times})$ is an adjoint pair, which implies

$$\mathrm{Hom}_{\mathbb{K}\text{-ALG}}(\mathbb{K}[G], A) \cong \mathrm{Hom}_{\mathrm{GROUP}}(G, A^{\times})$$

Thank you for your attention!

I hope that was of some help.