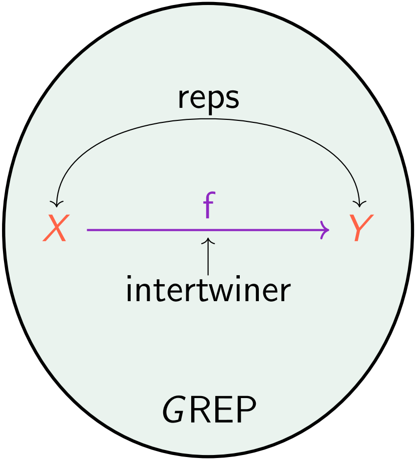
What are...intertwiners?

Or: Maps between representations

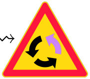
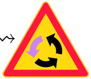
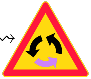
Equivariant maps



- Intertwiner = the correct maps between representations
- Invertible intertwiners give the notion of equivalence of representations
- Remains to find the "correct" notion ;-)

The same representations!?

$$\mathbb{Z}/3\mathbb{Z} = \{1, g, h\} \text{ with } g^2 = h, g^3 = 1$$

$\mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{C}\Big\{(1,0,0) \leftrightsquigarrow$  $,(0,1,0) \leftrightsquigarrow$  $,(0,0,1) \leftrightsquigarrow$  $\Big\}$

$$g \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad h \leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbb{Z}/3\mathbb{Z} \text{ acts on } \mathbb{C}\{(1,1,1),(0,1,0),(0,0,1)\}$$

$$g \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad h \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- These representation are the same up to base change
- We want to identity them, so we need base change as allowed maps

Base change

$$\begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\phi'_g} & \mathbb{C}^3 \\ \downarrow P & & \downarrow P \\ \mathbb{C}^3 & \xrightarrow[\phi_g]{} & \mathbb{C}^3 \end{array}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\phi_g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \phi'_g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

- Changing bases satisfies the above diagram (similarly for 1 and h)
- An intertwiner should be a generalized base change
- Take the above diagram as the definition of an intertwiner

For completeness: A formal definition

An intertwiner $f\colon V \to W$ between G-representations V, W is a map such that:

- f is linear
- f satisfies the commutative "base change" diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_g^V} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[\phi_g^W]{} & W \end{array}$$

  for all $g \in G$

Intertwiners also come under different names, *e.g.*

- Maps of representations
- Morphisms of representations
- G-equivariant maps/morphisms

Intertwiners are special matrices

$$S_2 = \{1, s\} \text{ acts on } V = \mathbb{C}\{(1,0),(0,1)\}$$

$$1 \leftrightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s \leftrightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{End}_{\mathbb{C}\mathsf{VECT}}(V) = \mathbb{C}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\begin{array}{ccc} V & \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)]{} & V \end{array} \rightsquigarrow \mathrm{End}_{S_2\mathsf{REP}}(V) = \mathbb{C}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

- Linear maps: $\dim \mathrm{End}_{\mathbb{C}\mathsf{VECT}}(V) = 4$ All matrices
- Intertwiners: $\dim \mathrm{End}_{S_2\mathsf{REP}}(V) = 2$ Smaller!

Thank you for your attention!

I hope that was of some help.