

**What is...a simple representation?**

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Or: The elements!

# Elements and simple representations

Chemistry	Group theory	Rep theory
Matter	Groups	Reps
Elements	Simple groups	Simple reps
Simpler substances	Jordan–Hölder theorem	Jordan–Hölder theorem
Periodic table	Classification of simple groups	Classification of simple reps

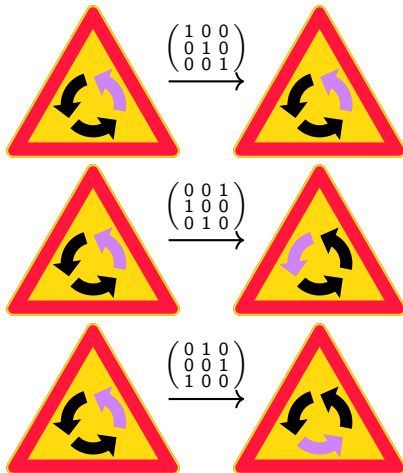
- ▶ **Question** What are the **simplest possible** representations?
- ▶ Whatever these are, they should play the role of elements in rep theory!

ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,  
ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

		Ti=50	Zr= 90	?=180.	
		V=51	Nb= 94	Ta=182.	
		Cr=52	Mo= 96	W=186.	
		Mn=55	Rh=104,4	Pt=197,1.	
		Fe=56	Ru=104,4	Ir=198.	
		Ni=Co=59	Pd=106,6	Os=199.	
H=1		Cu=63,4	Ag=108	Hg=200.	
	Be= 9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,3	?=68	U=116	Au=197?
	C=12	Si=28	?=70	Sn=118	
	N=14	P=31	As=75	Sb=122	Bi=210?
	O=16	S=32	Se=79,4	Te=128?	
	F=19	Cl=35,4	Br=80	I=127	
Li=7	Na=23	K=39	Rb=85,4	Cs=133	Tl=204.
		Ca=40	Sr=87,6	Ba=137	Pb=207.
		?=45	Ce=92		
		?Er=56	La=94		
		?Yt=60	Di=95		
		?In=75,6	Th=118?		

## Eigenvector = smaller representation

$\mathbb{Z}/3\mathbb{Z}$  acts on



- ▶  $(1, 1, 1)$  is an eigenvector of all matrices in the above representation  $\phi$
- ▶ The above is not an element: there is a simpler substructure  $\mathbb{C}\{(1, 1, 1)\}$

## Block decomposition

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$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

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- ▶ A base change verifies that  $\phi$  has two substructures
  - ▶ These are  $\mathbb{C}\{(1, 1, 1)\}$  and  $\mathbb{C}^2/\mathbb{C}\{(1, 1, 1)\}$
  - ▶  $\mathbb{C}\{(1, 1, 1)\} = \text{trivial}$

# For completeness: A formal definition

$\phi: G \rightarrow \text{GL}(V)$   $G$ -representation on a  $\mathbb{K}$ -vector space  $V$

- ▶ A  $\mathbb{K}$ -linear subspace  $W \subset V$  is  $G$ -invariant if  $G \cdot W \subset W$  **Substructure**
- ▶  $V \neq 0$  is called simple if  $0, V$  are the only  $G$ -invariant subspaces **Elements**

- ▶ Careful with different names in the literature:

$G$ -invariant  $\iff$  subrepresentation, simple  $\iff$  irreducible

- ▶ A crucial goal of representation theory

Find the periodic table of simple  $G$ -representations

Group	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1	1 H																		7 He
2	3 Li	4 Be											6 B	7 C	8 N	9 O	10 F	11 Ne	
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar	
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr	
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe	
6	55 Cs	56 Ba	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn	
7	87 Fr	88 Ra	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og	
			57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb			
			89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No			

$S_3$	(1)	(12)	(123)
$\chi_{\text{triv}}$	1	1	1
$\chi_{\text{sgn}}$	1	-1	1
$\chi_{\text{stand}}$	2	0	-1

## What happens for $\mathbb{Z}/3\mathbb{Z}$ ?

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$$Q = \begin{pmatrix} 1 & & 1 \\ 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) \\ 1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) \end{pmatrix}$$

$$Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}i(\sqrt{3} + i) & 0 \\ 0 & 0 & -\frac{1}{2}i(\sqrt{3} - i) \end{pmatrix}$$

$$Q^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}i(\sqrt{3} - i) & 0 \\ 0 & 0 & \frac{1}{2}i(\sqrt{3} + i) \end{pmatrix}$$

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- ▶  $\mathbb{Z}/3\mathbb{Z}$  has three simple representations over  $\mathbb{C}$ , one for each solution of  $X^3 = 1$
  - ▶ These are of dimension 1
  - ▶ Careful: simple  $G$ -representation need not to be of dimension 1!

**Thank you for your attention!**

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I hope that was of some help.