## **What is...a simple representation?**

Or: The elements!

# Elements and simple representations

| Chemistry | Group theory | Rep theory |
|---|---|---|
| Matter | Groups | Reps |
| Elements | Simple groups | Simple reps |
| Simpler substances | Jordan–Hölder theorem | Jordan–Hölder theorem |
| Periodic table | Classification of simple groups | Classification of simple reps |

- Question What are the simplest possible representations?
- Whatever these are, they should play the role of elements in rep theory!

**ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,**

ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

| | | | | | |
|---|---|---|---|---|---|
| | | | Ti=50 | Zr=90 | ?=180. |
| | | | V=51 | Nb=94 | Ta=182. |
| | | | Cr=52 | Mo=96 | W=186. |
| | | | Mn=55 | Rh=104,4 | Pt=197,1. |
| | | | Fe=56 | Ru=104,4 | Ir=198. |
| | | | Ni=Co=59 | Pd=106,6 | Os=199. |
| H=1 | | | Cu=63,4 | Ag=108 | Hg=200. |
| | Be=9,4 | Mg=24 | Zn=65,2 | Cd=112 | |
| | B=11 | Al=27,3 | ?=68 | Ur=116 | Au=197? |
| | C=12 | Si=28 | ?=70 | Sn=118 | |
| | N=14 | P=31 | As=75 | Sb=122 | Bi=210? |
| | O=16 | S=32 | Se=79,4 | Te=128? | |
| | F=19 | Cl=35,5 | Br=80 | I=127 | |
| Li=7 | Na=23 | K=39 | Rb=85,4 | Cs=133 | Tl=204. |
| | | Ca=40 | Sr=87,6 | Ba=137 | Pb=207. |
| | | ?=45 | Ce=92 | | |
| | | ?Er=56 | La=94 | | |
| | | ?Yt=60 | Di=95 | | |
| | | ?In=75,6 | Th=118? | | |

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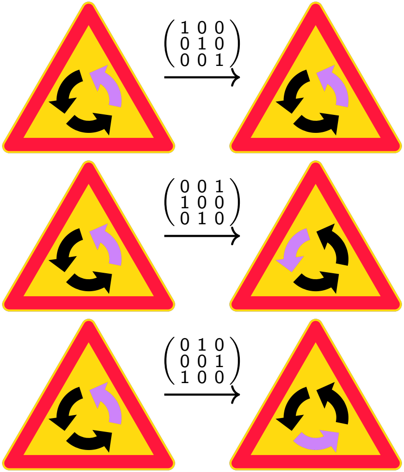
## Eigenvector = smaller representation

$\mathbb{Z}/3\mathbb{Z}$ acts on




- $(1,1,1)$ is an eigenvector of all matrices in the above representation $\phi$
- The above is not an element: there is a simpler substructure $\mathbb{C}\{(1,1,1)\}$

## Block decomposition

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- A base change verifies that $\phi$ has two substructures
- These are $\mathbb{C}\{(1,1,1)\}$ and $\mathbb{C}^2/\mathbb{C}\{(1,1,1)\}$
- $\mathbb{C}\{(1,1,1)\} =$ trivial

# For completeness: A formal definition

$\phi\colon G \to \mathrm{GL}(V)$ $G$-representation on a $\mathbb{K}$-vector space $V$

- A $\mathbb{K}$-linear subspace $W \subset V$ is $G$-invariant if $G \,.\, W \subset W$ Substructure
- $V \neq 0$ is called simple if $0, V$ are the only $G$-invariant subspaces Elements

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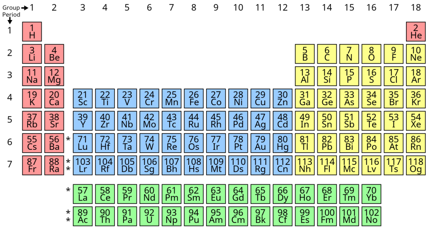
- Careful with different names in the literature:

$G$-invariant $\leftrightsquigarrow$ subrepresentation, simple $\leftrightsquigarrow$ irreducible

- A crucial goal of representation theory

Find the periodic table of simple $G$-representations




| $S_3$ | (1) | (12) | (123) |
|---|---|---|---|
| $\chi_{triv}$ | 1 | 1 | 1 |
| $\chi_{sgn}$ | 1 | −1 | 1 |
| $\chi_{stand}$ | 2 | 0 | −1 |

## What happens for $\mathbb{Z}/3\mathbb{Z}$?

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2}\left(-1 - i\sqrt{3}\right) & \frac{1}{2}\left(-1 + i\sqrt{3}\right) \\ 1 & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & \frac{1}{2}\left(-1 - i\sqrt{3}\right) \end{pmatrix}$$

$$Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}i\left(\sqrt{3} + i\right) & 0 \\ 0 & 0 & -\frac{1}{2}i\left(\sqrt{3} - i\right) \end{pmatrix}$$

$$Q^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}i\left(\sqrt{3} - i\right) & 0 \\ 0 & 0 & \frac{1}{2}i\left(\sqrt{3} + i\right) \end{pmatrix}$$

- $\mathbb{Z}/3\mathbb{Z}$ has three simple representations over $\mathbb{C}$, one for each solution of $X^3 = 1$
- These are of dimension 1
- Careful: simple $G$-representation need not to be of dimension 1!

**Thank you for your attention!**

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I hope that was of some help.