

What are...indecomposable representations?

Or: The elements! Well, kind of...

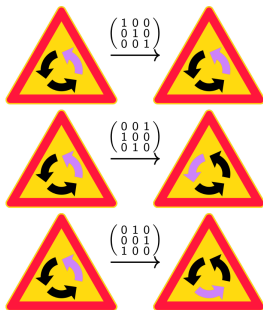
Other elements?

Group	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Period 1	1 H																	2 He
2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	* 71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	* 103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
			* 57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			* 89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

- ▶ Simple representations are the **no substructure elements**
- ▶ Indecomposable “block” representations are the **no decomposition elements**
- ▶ For complex G -representations these notions agree **Finite group miracle**

Simple \neq indecomposable

$\mathbb{Z}/3\mathbb{Z}$ acts on



$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

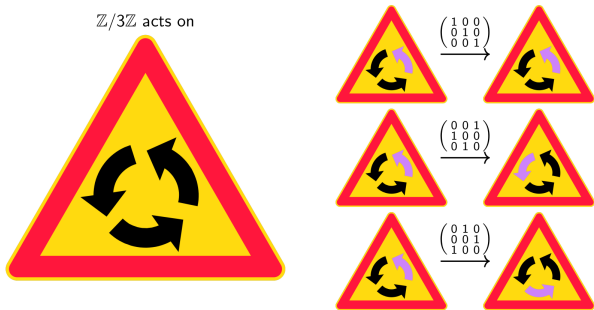
$$P^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- ▶ $(1, 1, 1)$ is an eigenvector regardless of \mathbb{K} **Substructure!**
- ▶ We have $\det(P) = 3$ and the block decomposition works only for $\text{char}(\mathbb{K}) \neq 3$
- ▶ In $\text{char}(\mathbb{K}) = 3$ the triangle representation

is indecomposable but not simple

A Jordan block



► Jordan decomposition over \mathbb{C} gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{change}]{\mathbb{C} \text{ base}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}$$

► Jordan decomposition over \mathbb{F}_3 gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{change}]{\mathbb{F}_3 \text{ base}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

For completeness: A formal definition

$\phi: G \rightarrow \text{GL}(V)$ G -representation on a \mathbb{K} -vector space V

► A \mathbb{K} -linear decomposition is $V \cong W \oplus X$ for G -invariant W, X **Blocks**

► $V \neq 0$ is called indecomposable if $V \cong W \oplus X$ implies $W = 0$ or $X = 0$ **Elements**

► In general we have

simple \Rightarrow indecomposable, simple $\not\Leftarrow$ indecomposable

► Simple \Leftrightarrow only trivial triangular blocks

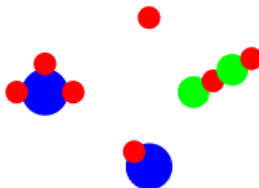
$$P^{-1}D(a)P = \begin{pmatrix} D^{(11)}(a) & D^{(12)}(a) \\ 0 & D^{(22)}(a) \end{pmatrix}$$

► Indecomposable \Leftrightarrow only trivial diagonal blocks

$$P^{-1}D(a)P = \begin{pmatrix} D^{(1)}(a) & 0 & \cdots & 0 \\ 0 & D^{(2)}(a) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{(k)}(a) \end{pmatrix} = D^{(1)}(a) \oplus D^{(2)}(a) \oplus \cdots \oplus D^{(k)}(a)$$

Elements? Maybe not quite...

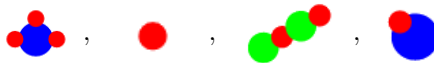
representations \leftrightarrow matter



simples \leftrightarrow elements



indecomposables \leftrightarrow compounds



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- ▶ Wouldn't "simple=indecomposable" be weird?
 - ▶ Well, it is true for complex representations of finite groups

Finite group miracle

Thank you for your attention!

I hope that was of some help.