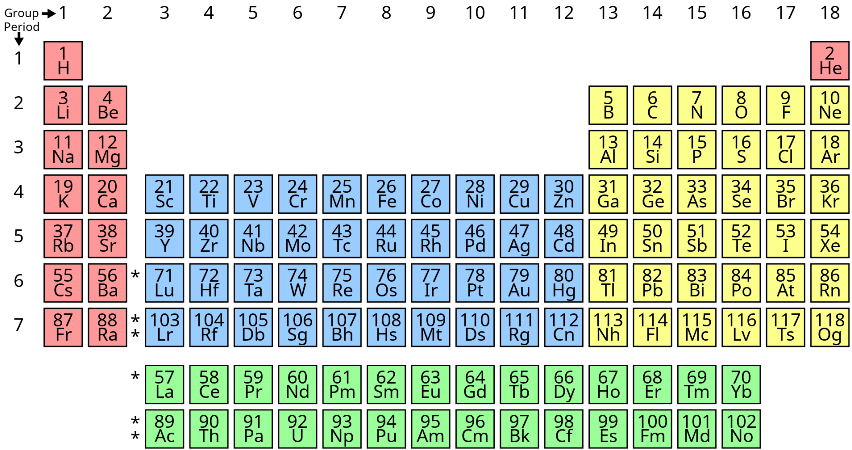
What are...indecomposable representations?

Or: The elements! Well, kind of...

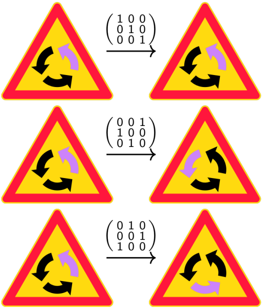
Other elements?



- Simple representations are the **no substructure elements**
- Indecomposable “block” representations are the **no decomposition elements**
- For complex G-representations these notions agree **Finite group miracle**

Simple $\neq$ indecomposable

$\mathbb{Z}/3\mathbb{Z}$ acts on



$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

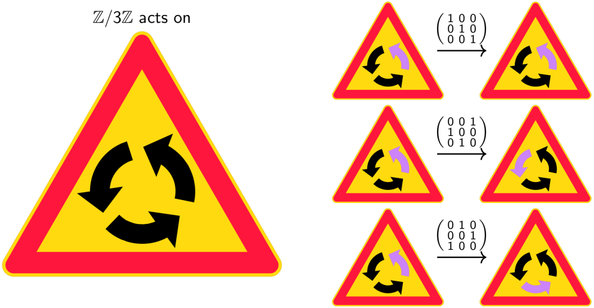
$$P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- $(1,1,1)$ is an eigenvector regardless of $\mathbb{K}$ Substructure!
- We have $\det(P) = 3$ and the block decomposition works only for $\mathrm{char}(\mathbb{K}) \neq 3$
- In $\mathrm{char}(\mathbb{K}) = 3$ the triangle representation

is indecomposable but not simple

A Jordan block

$\mathbb{Z}/3\mathbb{Z}$ acts on



- Jordan decomposition over $\mathbb{C}$ gives

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\xrightarrow[\text{change}]{\mathbb{C}\text{ base}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&e^{2\pi i/3}&0\\0&0&e^{4\pi i/3}\end{pmatrix},\begin{pmatrix}1&0&0\\0&e^{4\pi i/3}&0\\0&0&e^{2\pi i/3}\end{pmatrix}$$

- Jordan decomposition over $\mathbb{F}_3$ gives

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\xrightarrow[\text{change}]{\mathbb{F}_3\text{ base}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}1&2&1\\0&1&2\\0&0&1\end{pmatrix}$$

For completeness: A formal definition

$\phi\colon G \to \mathrm{GL}(V)$ G-representation on a $\mathbb{K}$-vector space V

- A $\mathbb{K}$-linear decomposition is $V \cong W \oplus X$ for G-invariant W, X Blocks
- $V \neq 0$ is called indecomposable if $V \cong W \oplus X$ implies $W = 0$ or $X = 0$ Elements

- In general we have

simple $\Rightarrow$ indecomposable, simple $\not\Leftarrow$ indecomposable

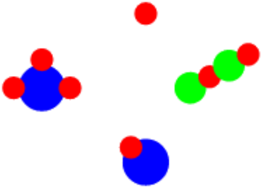
- Simple $\leftrightsquigarrow$ only trivial triangular blocks

$$P^{-1}D(a)P = \begin{pmatrix} D^{(11)}(a) & D^{(12)}(a) \\ 0 & D^{(22)}(a) \end{pmatrix}$$

- Indecomposable $\leftrightsquigarrow$ only trivial diagonal blocks

$$P^{-1}D(a)P = \begin{pmatrix} D^{(1)}(a) & 0 & \cdots & 0 \\ 0 & D^{(2)}(a) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{(k)}(a) \end{pmatrix} = D^{(1)}(a) \oplus D^{(2)}(a) \oplus \cdots \oplus D^{(k)}(a)$$

Elements? Maybe not quite...

representations $\leftrightsquigarrow$ matter 

simples $\leftrightsquigarrow$ elements 

indecomposables $\leftrightsquigarrow$ compounds 

- Wouldn't "simple=indecomposable" be weird?
- Well, it is true for complex representations of finite groups

Finite group miracle

Thank you for your attention!

I hope that was of some help.